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MATHEMATICAL QUESTIONS,

WITH THEIR

SOLUTIONS.

FROM THE "EDUCATIONAL TIMES."

WITH MANY

Papers and Solutions not published in the "Educational Times."

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CORRIGENDA.

p. 28, line 24, *for* "harmonic" *read* "geometric."

p. 28, for the *correct* enunciation of Quest. 1755, see p. 111.

p. 56, line 18, *for* $\left(\frac{l-c}{l}\right)$ *read* $\left(\frac{l-c}{l}\right)^2$.

p. 77, line 18, *for* 44001 *read* 100000.

p. 105, line 25, *for* "coordinates" *read* "determinants."

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1755. Conics are described passing though two angular points of a triangle, and having one focus coinciding with the point of intersection of the perpendiculars from the angles on the opposite sides; show that the locus of the other focus is a conic...28, 111

1860. Two points being taken at random on the *perimeter* of a rectangle, find the chance of the distance between them being less than a given length.................... 67

1874. To apply Plucker's Theory to determine the foci of a Cartesian oval. 69

1880. If two points are taken at random within any closed convex contour, the probability (p) of their distance not exceeding a given length (ρ) which is small compared with the dimensions of the area, is approximately $p = \frac{\pi}{\Omega}\rho^2 - \frac{2L}{3\Omega^2}\rho^3$, where L = length of contour, and Ω = enclosed area.

2. If the points are taken within a closed convex *surface* (S) inclosing a volume (V), we shall have $p = \frac{4\pi}{3V}\rho^3 - \frac{\pi S}{4V^2}\rho^4$ 103

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1903. If normals drawn to a conic at the points A, B, C, D pass through a point O, and meet the curve again in the points A_1, B_1, C_1, D_1; prove that the five points O, A_1, B_1, C_1, D_1 lie on an equilateral hyperbola, touching the equilateral hyperbola through ABCD at O. 104

1912. If a and b be the points of contact, with a curve of the third class, of a double tangent; and if this tangent be intersected in m, n, p by the three tangents to the curve which can be drawn from any point in the plane, then $\frac{am\,.\,an\,.\,ap}{bm\,.\,bn\,.\,bp}=\frac{\rho_a}{\rho_b}$, where ρ_a, ρ_b are the radii of curvature at the points a and b 88

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1933. If α, β be the eccentric angles of two adjacent vertices of a polygon inscribed in a fixed ellipse, and if the polygon envelope a fixed confocal, then will $\Sigma\cos(\alpha-\beta)=$ a constant. ... 111

1962. Required the *characteristics* of the system of conics having five-pointic contact with a curve of order m and class n. 47

1964. An iron shell, whose outside and inside diameters were respectively 9·8 and 7 inches, was filled with gunpowder, and being discharged at an elevation of 45°, reached its greatest elevation just over a house half a mile distant, where it burst: find the weight of powder required to discharge it from the mortar, and the height above the house at which the shell burst (supposing the mean weight of 30 cubic inches of gunpowder to be 1 lb.) 27

1982. A cylindrical vessel containing fluid revolves round a vertical axis: required the form which the fluid will assume. If the vessel be full and be whirled round the same axis, find the whole pressure against the side, also find the whole pressure on the top of the cylinder. 46

1988. If N be a composite number, find the relation which must exist between the periods of the prime factors of N in order that the repetend of the recurring decimal which is equal to $\frac{1}{N}$ may be complementary. 31

1991. Given a point and three lines; it is required to draw through the point a plane meeting the three lines in three points equidistant from the given point. 33

1992. Construct geometrically (1) the four chords of contact with a given conic of the four inscribed conics which pass through three given points, (2) the eight planes of contact with a given quadric of the eight inscribed quadrics which pass through four given points.. 92

1993. If P is a point on a circle, in which A and B are fixed points on a diameter at equal distances from its centre, the curve

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envelope of lines cutting harmonically the two circles whose centres are A and B and radii AP, BP respectively, is independent of the position of P on the circle. 34

1995. Let G be the centre of gravity of any triangle ABC ; I the centre of the inscribed circle ; λ_1, λ_2, λ_3 the points where the sides BC, CA, AB are touched by the escribed circles opposite A, B, C, whose centres are I_1, I_2, I_3 ; Q the point of intersection of $A\lambda_1$, $B\lambda_1$, $C\lambda_3$; Q′ the point inverse to Q ; G′ the inverse of G ; then the polars of I, G′, Q′, with respect to the triangle, meet in the same point. Also, if these respective polars be denoted by P_1, P_2, P_3, and the harmonic conjugate of P_3, with respect to P_1, P_2, by P_4 ; show that the points of intersection of the circles (ABC), $(I_1I_2I_3)$ with P_3, P_4 respectively, lie on another circle which touches the circle (I). 80

2000. Let ABCD be a rectangle. Bisect one of the sides, BC, in E, and with centre E and distance EB describe a circle. From E draw the radius EG (towards AD) parallel to CD, and join GC. Then join BD, cutting GC (or, as the case may be, GC produced) in H. From H draw the straight line HK, parallel to EG, meeting the circle (towards AD) in K. Join BK and produce it to meet CD (or CD produced) in L : then CL shall be the side of a square equal to the rectangle ABCD. 61

2010. If Q, R be the foci of an ellipse inscribed in a triangle ABC, prove that
$$AQ\,.\,AR\,.\,BC+BQ\,.\,BR\,.\,CA+CQ\,.\,CR\,.\,AB = BC\,.\,CA\,.\,AB.$$ 43

2110. Prove that the locus of the foci of the parabolas which pass through three given points is a unicursal quintic curve passing through the two circular points at infinity 17

2112. To find a point P in a given circular arc ABC, so that the chords AP, BP drawn therefrom to the ends of the chord AB may be given or a maximum 21

2115. 1. The line joining any two of the centres A, B, C, D of four circles (A), (B), (C), (D) cutting each other at right angles in a plane, is perpendicular to the line joining the other two ; any circle (A) cuts the sides of the triangle BCD harmonically, and the square of its radius varies inversely as the area of the triangle BCD. The sum of the squares of the radii is equal to the square of the diameter of the equal circles circumscribing the triangles, and the sum of the inverse squares is nothing.

2. If the circle described on AB as a diameter be denoted by (AB), then the coaxal system of circles (A) (B) (AB) and the line CD are orthogonal to the coaxal system of circles (C) (D) (CD) and the line AB.

3. The locus of the real foci lying on lines parallel to AB of conics inscribed in the triangle formed from the intersection of the lines (BC, AD), (CA, BD) and (AB, CD) is the circle (CD) and the line AB. 47

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2118. Find the ratios $l : m : n$ from the equations

$$l^2 \cos 2\alpha + m^2 \cos 2\beta + n^2 \cos 2\gamma + \kappa \left\{ mn \cos (\beta + \gamma) + nl \cos (\gamma + \alpha) + lm \cos (\alpha + \beta) \right\} = 0,$$

$$l^2 \sin 2\alpha + m^2 \sin 2\beta + n^2 \sin 2\gamma + \kappa \left\{ mn \sin (\beta + \gamma) + nl \sin (\gamma + \alpha) + lm \sin (\alpha + \beta) \right\} = 0,$$

where $\kappa = \pm 2$ or -1. 26

2130. Find the average area of all the quadrilaterals which can be inscribed in a given circle. 30

2135. A tangent to an ellipse is a chord of a concentric circle, whose radius is equal to the distance between the ends of the axes of the ellipse; show that the straight lines which join the ends of the chord to the centre are conjugate diameters. 45

2210. A semi-ellipse is placed in a vertical position, and a heavy particle is just set in motion at the highest point of the curve, and descends down it by its own weight; required the point at which it will leave the curve, and its velocity at that point. ... 109

2220. A, B, C, D are four points on a given circle, and through every pair, as AB, another circle (AB) is drawn; then the pair of circles (AB), (CD) intersects the pair (AC), (DB) in four new points on a circle U; the pair (AC), (DB) meets (AD), (BC) on a circle V; and the pair (AD), (BC) meets (AB), (CD) on a circle W; also the three circles U, V, W have a common radical axis. (This may be extended to spheres; and these are also analoguous properties of rectangular hyperbolas.) ... 37

2223. Through the vertices A, B, C of a triangle ABC any three parallel straight lines AX, BY, CZ are drawn, cutting the sides BC, CA, AB in X, Y, Z; show that the envelope of the axis of homology of the triangles ABC, XYZ is the maximum ellipse that can be inscribed in the triangle ABC. 94

2225. A cone floats in a fluid, the densities of both being given; but a weight w being placed on the top of the cone, its draught in the fluid is n times its former draught; show that the weight (w') of the cone is $w' = \frac{w}{n^3 - 1}$. 29

2232. Prove that the axes of the parabolas which have a common focus, and pass through two given points, are parallel to the asymptotes of the hyperbola which passes through the common focus, and has the two given points for foci 49

2236. Prove that

(1.) $$\tan^{-1} x = \frac{x}{\sqrt{(1+x^2)}} \left\{ 1 + \frac{1}{2} \cdot \frac{1}{3} \left(\frac{x^2}{1+x^2} \right) + \frac{1 . 3}{2 . 4} \cdot \left(\frac{x^2}{1+x^2} \right)^2 + \&c. \right\};$$

(2.) $$\pi = \frac{28}{10} \left\{ 1 + \frac{2}{3} \cdot \frac{2}{10^2} + \frac{2 . 4}{3 . 5} \cdot \left(\frac{2}{10^2} \right)^2 + \frac{2 . 4 . 6}{3 . 5 . 7} \cdot \left(\frac{2}{10^2} \right)^3 + \&c. \right\}$$

$$+ \frac{30336}{10^5} \left\{ 1 + \frac{2}{3} \cdot \frac{12^2}{10^5} + \frac{2 . 4}{3 . 5} \cdot \left(\frac{12^2}{10^5} \right)^2 + \frac{2 . 4 . 6}{3 . 5 . 7} \cdot \left(\frac{12^2}{10^5} \right)^3 + \&c. \right\}$$ 62

2238. Find (1) the envelope, (2) the locus of centres, of circles cutting a given circle orthogonally, and intercepting a given length on a given straight line; also find (3) the envelope of the radical axis of the given and the orthogonal circles. 64

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2241. If C denote the circle which circumscribes any three given points α, β, γ; and T the four circles which touch the sides of the triangle formed by joining the points in question; prove that the two points common to C and T (*i.e.* the circular points at infinity) are the foci of the two imaginary parabolas which pass through the centres of T. Also show that the curve-locus of the foci of all parabolas which can be drawn through α, β, γ passes through the *ten* points where the circle C cuts the system T. 95

2242. If a surface of the second order circumscribe a tetrahedron, and cut another in two plane sections, and if one of these sections pass through a fixed point; find the envelope of the other. 23

2246. Let U be a homogeneous expression in n (any number of) variables of the second degree: let A_n be the discriminant: erasing from A_n the right-hand column and bottom row, call the determinant so formed A_{n-1}: let this process and notation be continued until we arrive at A_1 (which will be the coefficient of some one of the squares of the variables in U): prove that if every one of these determinants, from A_n to A_1, be positive, $U = 0$ cannot be satisfied by any real values of the variables. 105

2248. Prove that any tangent to a conic is divided harmonically by the point of contact and the three points where it meets any two tangents and their chord of contact. 42

2250. From the focal equation $x^2+y^2 = (lx+n)^2$, of a conic, deduce the remaining three focal equations. 26

2251. From an urn containing n counters, numbered 1, 2, 3,......n, k drawings are made, the counter drawn being replaced each time. If the highest number drawn be m, prove that the most probable value of n is the integer next above, or next below $\sqrt[k]{2} \,.\, (m-1)$ 23

2253. If four circles have a common radical centre, it is possible to find four planes which intersect, two and two, at angles equal to those at which the circles intersect, but not otherwise 22

2254. If f be any quantic in (x, y), and

$$\phi = \left(ax\frac{d}{dx} + ay\frac{d}{dx} + \beta x\frac{d}{dy} + by\frac{d}{dy}\right) f,$$

show that, if $(a-b)^2+4a\beta =$ or < 0, the number of real roots in ϕ is at least as great as in f. 37

2256. A quadrilateral in an ellipse has one diagonal a diameter of the curve, and the other a focal chord parallel to the conjugate diameter; show that its area varies as the conjugate diameter; also find the locus of the intersection of the diagonals 50

2258. Prove that

$$\frac{\pi^2}{8} = \frac{3^2}{2.4}\cdot\frac{5^2}{4.6}\cdot\frac{7^2}{6.8}\cdot\frac{11^2}{10.12}\cdot\frac{13^2}{12.14}\cdot\frac{17^2}{16.18}\cdots\cdots$$

the numerators being the squares of the prime numbers. 37

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2259. Prove *geometrically* that if A, B, C, D, E be five points on a circle, the consecutive intersections of the nine-point circles of the triangles ABC, BCD, CDE, DEA, EAB, lie on another circle whose radius is one-half that of the first. 57

2262. Evaluate the definite integral

$$\int_0^\pi \log \sin \theta \, . \, d\theta. \text{ ....} \quad 46$$

2263. Two chords are drawn at random in a circle; find the chance that they will intersect, supposing a random chord to mean any line joining two points of the circle, all lengths less than the diameter being equally probable, and two chords drawn at random to mean two such chords inclined at any angle, all inclinations being equally probable 20, 51

2265. A variable point determines with reference to a fixed cubic curve two covarying conics, viz., one passing through the six points of contact of tangents drawn from the point, and another passing through the six points where these tangents meet the curve again: prove that these two conics have always double contact with one another. The chord of contact is the line polar of the point with reference to the cubic. 22

2268. Find the values of $n_1, n_2, n_3 \dots n_k$ which make $\dfrac{n_1^2+n_2^2+n_3^2+\dots}{(a_1n_1+a_2n_2+\dots)^2}$ or $\dfrac{\Sigma n^2}{(\Sigma an)^2}$ a minimum, $n_1, n_2, n_3 \dots$ being connected by the equations $\Sigma bn = b_1n_1+b_2n_2+\dots = 0$, $\Sigma cn = c_1n_1+c_2n_3+\dots = 0$, &c. &c., the number of these equations being less than $k-1$. 20

2270. To reduce the equation of a bicircular quartic into the form $SS'-k^3L=0$, where $S=0$, $S'=0$ are the equations of two circles, $L=0$ the equation of a line. (See Salmon's *Higher Plane Curves*, p. 128) 52

2271. A, B, C, D are four successive points in a straight line. If (I) two points be taken at random in AD, and one point at random in BC, find the chance that the latter will lie between the two former. And again, if (II) two points be taken at random in BC, and one point at random in AD, find the chance that the latter will lie between the two former.......... 38

2272. Through each angle of a triangle, and through the intersection of its three perpendiculars, a circle is drawn. Prove that if the tangents to the three circles at the angles of the triangle be parallel to one another, the circles will intersect in a second point. Find the locus of the latter point when the common direction of the three tangents varies. 24

2273. L'équation $1+ax+bx^2+\dots=0$ ayant toutes ses racines réelles, si l'on pose $(1+ax+bx^2+\dots)^{-1} = 1+Ax+Bx^2+\dots$, et qu'on opère avec la série ainsi obtenue sur une équation quelconque $F(x)=0$, la nouvelle équation, savoir $F(x)+AF'(x)+BF''(x)+\dots=0$, aura au moins autant de racines imaginaires que la première. 53

2274. If p_{kl} denote the perpendicular from K on the polar of L

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relative to a given conic, show that the equation in areal coordinates of the conic may be written

$$\frac{p_{aa}}{p_{oa}}x^2+\frac{p_{bb}}{p_{ob}}y^2+\frac{p_{cc}}{p_{oc}}z^2+\frac{2p_{bc}}{p_{oc}}yz+\frac{2p_{ca}}{p_{oa}}zx+\frac{2p_{ab}}{p_{ob}}xy=0,$$

the vertices of the triangle of reference and the centre being respectively A, B, C, O. 54

2277. Through a point O straight lines AO, BO, CO are drawn to meet the sides of the triangle ABC in D, E, F. Find the locus of O in order that the three perpendiculars from A, B, C on EF, FD, DE respectively may meet in a point.................. 39

2278. Supposing the probability of a point taken at hazard in space falling within the element of space dS to be $\left(\frac{c}{\pi}\right)^{\frac{3}{2}}\epsilon^{-cr^2}dS$, where r is the distance of dS from a fixed point O, and c some positive constant; determine the probability that the triangle formed by O, and two points taken at hazard in space, shall be acute-angled. 55

2282. Draw the minimum chord of a given angle which can be cut in a given ratio by a given line.................................... 41

2284. (1.) Q is the point of intersection of the polars of any point P with respect to two given similar and similarly situated conics; show that the locus of the middle point of the line PQ is the radical axis of the curves in question. Hence, (2) if D, E, F be the feet of the perpendiculars of any triangle ABC; and AL, BM, CN be lines drawn through A, B, C parallel to EF, FD, DE, and cutting BC, CA, AB in the points L, M, N; it is required to prove that the axis of homology of the triangles ABC, DEF passes through the middle points of the segments AL, BM, CN.................................... 78

2286. If we have $(n-2)$ sets of n quantities each, $(a_1, a_2 \ldots a_n)$, $(\beta_1, \beta_2 \ldots \beta_n)$, $\ldots(\lambda_1, \lambda_2, \ldots \lambda_n)$, connected with the n quantities $(r_1, r_2 \ldots r_n)$ by $\frac{1}{2}n\ (n-1)$ equations of which the type form is

$$(a_k-a_l)^2+(\beta_k-\beta_l)^2+\ldots(\lambda_k-\lambda_l)^2=r_k^{\,2}+r_l^{\,2};$$

then show that $\frac{1}{r_1^{\,2}}+\frac{1}{r_2^{\,2}}+\ldots+\frac{1}{r_n^{\,2}}=0$ and $\frac{\mathrm{P}}{r_1^{\,2}}+\frac{\mathrm{P}_2}{r_1^{\,2}}+\ldots+\frac{\mathrm{P}_n}{r_n^{\,2}}=0$, where P is any one of the quantities $a, \beta, \gamma \ldots \lambda$.................... 99

2291. Prove that

$$\frac{[-\log(1+x)]^n}{\Pi n}=\frac{x^n}{\Pi n}+\mathrm{S}_{n,1}\frac{x^{n+1}}{\Pi(n+1)}+\mathrm{S}_{n+1,2}\frac{x^{n+2}}{\Pi(n+2)}$$

$$+\mathrm{S}_{n+2,3}\frac{x^{n+3}}{\Pi(n+3)}+\ldots\ \text{i.e.}=\sum_{\mu=\infty}^{\mu=0}\left(\mathrm{S}_{n+\mu-1,\mu}\frac{x^{n+\mu}}{\Pi(n+\mu)}\right),$$

where $\mathrm{S}_{i,j}$ means the arm of the j-ary products of the numbers $1, 2, 3, \ldots i$. 72

2294. From one end (A) of the major axis of an ellipse draw any chord AB, and through the centre O draw OP parallel to the tangent at B, and meeting AB in P. Find the locus and area of the curve traced out by P.................................... 59

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2296. To express the values of $\frac{1}{4}\pi$ and $\log_e$ (2) in the form of continued fractions. 79

2297. If two points are taken at random in a given line of length l, show that the probability of no one of the three segments exceeding a given length $c\ (>\frac{1}{3}l)$ is

(1) if $c<\frac{1}{2}l,\ p_1=\left(\frac{3c-l}{l}\right)^2$; (2) if $c>\frac{1}{2}l,\ p_2=1-3\left(\frac{l-c}{l}\right)^2$ 56

2298. If r_1, r_2 be a pair of focal *covectors* of an ellipse, and ϕ_1 the angle between them, then prove that

$r_2(1-\cos\phi_1)=p$, where p is the semi-parameter.

Def.—By *covectors* I mean a system of vectors so taken that the nth is parallel to the tangent at the extremity of the $(n-1)$th. 42

2299. It is a known property, in curves of the second order, that, at any point of a conic, the chord of curvature, which coincides in direction with either of the two focal lines through the point, is equal in length to either of the two focal chords of the curve whose directions are parallel to the tangent at the point.

Show, analogously, in surfaces of the second order, that, at any point of a quadric, the chord of normal curvature, which coincides in direction with any one of the four bifocal lines through the point, is equal in length to any one of the four bifocal chords of the surface whose directions are parallel to the corresponding tangent at the point. 93

2300. Show that 24 cusped cubic curves can usually be drawn through 7 given points. 24

2302. Let A, B, C denote the vertices of a given triangle ABC; Q, R the opposite extremities of any diameter of the circle circumscribing it; then it is required to prove that the centres of the five equilateral hyperbolas which pass through A, B, C, Q, R, taken four and four together, all lie on the nine-point circle of the triangle. 96

2303. If $u=\log(x^3+y^3+z^3-3xyz)$, prove that

$$\Sigma\,.\left\{\frac{d^n u}{dx^{\alpha+1}\,.\,dy^{\beta}\,.\,dz^{\gamma}}+\frac{d^n u}{dx^{\alpha}\,.\,dy^{\beta+1}\,.\,dz^{\gamma}}+\frac{d^n u}{dx^{\alpha}\,.\,dy^{\beta}\,.\,dz^{\gamma+1}}\right\}=\frac{3n(n-1)\,.\,\underline{|n-1}\,.\,(-1)^{n-1}}{2(x+y+z)^n},$$

where α, β, γ are subject to the condition $\alpha+\beta+\gamma=n-1$, and have *each* of them all integral values from 0 to $(n-1)$ consistent with this condition; also Σ signifies the sum of all these values 58

2305. Compare the greatest and least inscribed squares in a scalene triangle. 28

2306. Eliminate θ between the following equations:—

$x\cos(\theta-A)+y\cos\theta=2R\sin(\theta+C)\cos\theta\cos(\theta-A)$. .... (1),

$x\sin(\theta-A)+y\sin\theta=2R\left\{\sin(\theta+C)\sin\theta\cos(\theta-A)-\cos B\cos\theta\right\}$ (2),

where A, B, C are the angles of a plane triangle 60

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2309. Show that for n things
$1-$(no. of partitions into 2 parts) $+ 1.2$ (no. of partitions into 3 parts) ...... $\pm 1.2.3 \ldots (n-1)$ (no. of partitions into n parts) $= 0$ 87

2315. Sum the series
$$(1) \ldots\ldots \frac{x}{1} - \frac{x^4}{4} + \frac{x^7}{7} - \&\text{c.}, \quad (2) \ldots\ldots \frac{x}{1} - \frac{x^5}{5} + \frac{x^9}{9} - \&\text{c.} \ldots$$ 44

2320. Draw a circle and its involute, and find the locus of the centre of a circle which cuts both curves orthogonally.................. 109

2326. If three points be taken at random on the surface of a sphere, find the respective chances that the triangle they determine will have (1) all its angles *acute*, (2) all its angles *obtuse*....... 105

2328. (1) Let G be the centre of gravity of any triangle ABC; β and γ the middle points of two of its sides (AB, AC, say); show that the circles described upon the lines AG, $\beta\gamma$, BC as diameters, are coaxal. Hence (2) if P be the point of intersection of the perpendiculars of the triangle; α the middle point of the side BC; g a point on αP such that $\alpha g = \frac{1}{3}\alpha P$; also α' a point on Aα produced, such that $A\alpha' = 2A\alpha$; then the nine-point circle of the triangle is coaxal with the circles described upon Ag and Pα' as diameters. 76

2329. Let ABC be a triangle; on the *same* side of AB as C draw RP parallel to AB, and at a distance from it equal to AC; also on the *opposite* side of AC to B, draw PQ parallel to AC, and at a distance from it equal to AB; join AP, then AP is parallel to BC.. 64

2331. Show that it is possible to find (X, Y, Z) linear functions of the trilinear coordinates (x, y, z) such that the equations $x\text{X} = y\text{Y} = z\text{Z}$ may determine four given points. 106

2332. Show that but two Cartesians can be drawn having a given axis and passing through any four given points, and prove thereby, with the aid of the proposition in Quest. 1990, that all the Cartesians which can be drawn through four given concyclic points consist exclusively of two tribes (*families of families*), whose foci lie respectively on the two circular cubics, of which the four given points are foci. 74

2335. Sum the series
$$(1) \ldots \frac{1}{1.4} + \frac{1}{5.8} + \frac{1}{9.12} + \&\text{c.}, \quad (2) \ldots \frac{1}{1.3} + \frac{1}{7.9} + \frac{1}{13.15} + \&\text{c.}$$ 44

2338. ABC is a triangle, D, E, F are points in the sides BC, CA, AB respectively, such that
$$\text{BD} : \text{DC} = \text{CE} : \text{EA} = \text{AF} : \text{FB} = l : m.$$
Find the area of the triangle formed by the three lines AD, BE, CF. 89

2347. No real straight line can be drawn parallel to the imaginary straight line whose equation in trilinear coordinates is
$$l\alpha + m\beta + n\gamma + i\,(l'\alpha + m'\beta + n'\gamma) = 0, \text{ (where } i = \sqrt{-1}),$$
unless
$$\begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ a, & b, & c \end{vmatrix} = 0$$ 110

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2363. Prove that, if n be odd and > 1,
$$\left\{\frac{\Delta}{1} - \frac{r+1}{2r+1} \cdot \frac{\Delta}{2} + \frac{(r+1)(r+2)}{(2r+1)(2r+2)} \cdot \frac{\Delta^3}{3} + \&c. \right\} 0^n = 0 \;\ldots\ldots\ldots\; 91$$

2365. Into how many distinct parts is (1) an infinite line divided by n points, (2) infinite space of two dimensions by n straight lines, and (3) infinite space of three dimensions by n planes? 107

2373. What is the condition that a circle should cut a conic orthogonally? How many circles can be drawn through two points to cut a given conic orthogonally? 108

2377. If the distance between the foci of an ellipse subtend angles ϕ, ψ at the extremities of a pair of semiconjugate diameters, then $\tan^2 \frac{1}{2}\phi + \tan^2 \frac{1}{2}\psi = \frac{a^2e^2}{b^2}$. 111

2385. If the circumference of a circle be divided into any four arcs, and these arcs be bisected, prove that the chords joining the points of bisection intersect at right angles.......................... 110

Unsolved Questions.

1971. (Proposed by C. W. Merrifield, F.R.S.)—In a given tetrahedron, to find a set of rectangular coordinate axes, such that each axis shall pass through a pair of opposite edges.

1977. (Proposed by M. W. Crofton, B.A.)—If three lengths are taken at random, find the chance that it will be possible to form a triangle with them; the extreme limit, whatever its magnitude, being supposed the same for all three.

1980. (Proposed by W. S. Burnside, M.A.)—If the sides ad, bd, cd of a tetrahedron $abcd$ meet a surface of the second order in the points a, a_1; b, b_1; c, c_1, respectively: prove that the sections of the surface by the planes bca_1, cab_1, abc_1, abc are all touched by another plane section of the surface.

1998. (Proposed by R. Ball, M.A.)—If k be the constant term in the equation of a surface, and $\Delta = 0$ the condition necessary that this surface and three others pass through a point, what is the geometrical meaning of the roots of the equation $e^{-x\frac{d}{dk}}\Delta = 0$.

2119. (Proposed by Matthew Collins, B.A.)—The equation $Ax^7 - 29 = 547y$ is frequently impossible when A, x, and y are whole numbers (*ex. gr.*, it is so when A = 1, 2, 3, 4, or 10, 20, 30, ... 90); required the chance of its possibility when A is taken at random equal to a whole number less than 100.

2221. (Proposed by R. Ball, M.A.)—1. Determine the locus of the feet of perpendiculars from a fixed point on the generating lines of a system of confocal quadrics.

2. Prove that this locus is described by foci of the plane sections of the system, which pass through the fixed point.

No.

2228. (Proposed by Dr. Wilson.)—The axes of two equal cylinders intersect at right angles; find the attraction of the intercepted solid upon a particle situated at the intersection of the four curved edges of the solid.

2244. (Proposed by W. Lea.)—Form 11 symbols into sets, 5 symbols in a set, so that every combination of 4 symbols shall appear once in the sets.

2247. (Proposed by Felsineus.)—Etudier les cas de réalité et d'imaginairété des quatre cones qui font partie d'un faisceau de surfaces du second ordre.

2255. (Proposed by M. Gardiner.)—If all the sides but one of a variable polygon inscribed in a fixed surface of the second order pass in the same order through an arbitrary system of fixed points in space, the free side sometimes passes through a fixed point, sometimes intersects two fixed lines reciprocal polars with respect to the surface, and sometimes envelopes a second surface of the second order, having double contact with the original surface; investigate the circumstances under which these particulars respectively take place.

2261. (Proposed by S. Bills.)—Sixty-three school-boys walk out three abreast for thirty-one days in succession; it is required to arrange them daily so that no two of them shall walk twice abreast. The *actual* arrangement is required to be exhibited.

2264. (Proposed by W. B. Davis, B.A.)—In how many ways can a polygon of n sides be divided into triangles by diagonals, (1) when the polygons are convex, (2) when some of the angles are re-entrant?

2281. (Proposed by W. K. Clifford, B.A.)—Six circles pass through twelve points on a conic in the following order,

$$(a)\ldots A_1A_2A_3A_4,\quad (b)\ldots B_1B_2B_3B_4,\quad (c)\ldots C_1C_2C_3C_4;$$
$$(d)\ldots A_1A_2B_3C_4,\quad (e)\ldots B_1B_2C_3A_4,\quad (f)\ldots C_1C_2A_3B_4;$$

prove that two circles and another point may be taken arbitrarily, and that the circles *abc* meet the circles *def* in six new points which lie on the circumference of another circle.

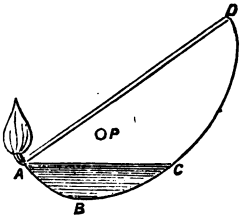
2317. (Proposed by Professor Sylvester.)—1. Four points are taken at random upon a finite straight line; find the probability that it may be possible to form with the three segments between them an acute-angled triangle.

2. Two points are taken at random upon a straight line AB, and two others upon BC lying in AB produced; find the probability that it may be possible to form with the three segments included between these four points an acute-angled triangle.

2325. (Proposed by Professor Sylvester.)—Prove that the locus of the foci of all the conics that can be described about a given quadrilateral is a curve of the sixth order, of which the circular points at infinity are double points, and the angles of the quadrilateral *duplicate* foci. [By a *duplicate* focus is to be understood a point the lines joining which to the circular points at infinity are each of them double tangents to a curve.]

No.
2350. (Proposed by W. S. B. Woolhouse, F.R.A.S.)—Determine the average area of all triangles drawn on the surface of a given rectangle which have one angle at a given point.

2353. (Proposed by Professor De Morgan.)—The late Dr. Milner, President of Queen's College, Cambridge, constructed a lamp, which General Perronet Thompson remembers to have seen. It is a thin cylindrical bowl, revolving about an axis at P, and the curve ABCD is such that, whatever quantity of oil ABC may be in the bowl, the position of equilibrium is such that the oil just wets the wick at A. What is the curve ABCD?



2355. (Proposed by the Rev. W. Roberts, M.A.)—The equation of the second negative pedal of an ellipse, with reference to the focus as origin, is found by equating to zero the discriminant of the following cubic in θ,

$$(2a+2c-\theta)(2a-2c-\theta)\,\theta+c\theta x+(a\theta-b^2)\sqrt{(x^2+y^2)}=0,$$

a, b being the semi-axes, and c the distance of the focus from the centre.

2356. (Proposed by Capt. Clarke, R.E., F.R.S.)—A tetrahedron is thrown into the air and a small shot fixed through it; show that the chance of one of the faces (which is marked) being perforated, is $2F \div S$, where F is the area of the marked face and S the whole surface of the tetrahedron.

2358. (Proposed by Professor Cremona.)—La condition qu'une conique divise harmoniquement les trois diagonales d'un quadrilatère circonscrit à une autre conique, coïncide avec la condition que la première conique soit circonscrite à un triangle conjugué à l'autre.

2360. (Proposed by N'Importe.)—Find the average of the areas of all the triangles in which there are given (1) an angle and the sum of the two containing sides, (2) the sum of two sides only, (3) one side and the sum of the other two, (4) the sum of the three sides.

2378. (Proposed by N'Importe.)—Find the average (1) of the areas of all the circles which can be drawn within the circumference of a given circle, and (2) of the volumes of all the spheres which can be drawn within the surface of a given sphere; also find the probability (3) that any one of the circles taken at random in (1) will be less than a given area, and (4) that any one of the spheres taken at random in (2) will be less than a given volume.

MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

2110. (Proposed by Professor CAYLEY.)—Prove that the locus of the foci of the parabolas which pass through three given points is a unicursal quintic curve passing through the two circular points at infinity.

I. *Solution by the* PROPOSER.

More generally it may be shown that for the conics which pass through three given points and touch a given line, the locus of the intersection of the tangents drawn from two fixed points Q, Q′ on this line to each conic of the series is a unicursal quintic passing through the two points Q and Q′.

Taking the three given points to be the angles of the triangle ($x = 0,\ y = 0,\ z = 0$), and the points Q, Q′ to be the points (α, β, γ) and $(\alpha', \beta', \gamma')$ respectively, the equation of a conic through the three points is

$$fyz + gzx + hxy = 0,$$

which conic will touch the line through the points (α, β, γ) $(\alpha', \beta', \gamma)$, if

$$\sqrt{\{f(\beta\gamma' - \beta'\gamma)\}} + \sqrt{\{g(\gamma\alpha' - \gamma'\alpha)\}} + \sqrt{\{h(\alpha\beta' - \alpha'\beta)\}} = 0.$$

The equation of the pair of tangents from (α, β, γ) to the conic is

$$(f^2, g^2, h^2, -gh, -hf, -fg)(\gamma y - \beta z, \alpha z - \gamma x, \beta x - \alpha y)^2 = 0,$$

that is

$$\begin{aligned} x^2(g\gamma + h\beta)^2 + y^2(h\alpha + f\gamma)^2 + z^2(f\beta + g\alpha)^2 \\ + 2yz\left\{2gh\alpha^2 - (h\alpha + f\gamma)(f\beta + g\alpha)\right\} \\ + 2zx\left\{2hg\beta^2 - (f\beta + g\alpha)(g\gamma + h\beta)\right\} \\ + 2xy\left\{2fg\gamma^2 - (g\gamma + h\beta)(h\alpha + f\gamma)\right\} = 0, \end{aligned}$$

but one of the tangents through (α, β, γ) being

$$x(\beta\gamma' - \beta'\gamma) + y(\gamma\alpha' - \gamma'\alpha) + z(\alpha\beta' - \alpha\beta) = 0,$$

it follows that the other tangent is

$$x\frac{(g\gamma + h\beta)^2}{\beta\gamma' - \beta'\gamma} + y\frac{(h\alpha + f\gamma)^2}{\gamma\alpha' - \gamma'\alpha} + z\frac{(f\beta + g\alpha)^2}{\alpha\beta' - \alpha'\beta} = 0.$$

Hence, writing for shortness

$$\mathrm{A} = g\gamma + h\beta,\quad \mathrm{B} = h\alpha + f\gamma,\quad \mathrm{C} = f\beta + g\alpha,$$

$$\mathrm{A}' = g\gamma' + h\beta',\quad \mathrm{B}' = h\alpha' + f\gamma',\quad \mathrm{C}' = f\beta' + g\alpha',$$

the equations of the tangents from Q, Q′ respectively are

$$\mathrm{A}^2 \frac{x}{\beta\gamma' - \beta'\gamma} + \mathrm{B}^2 \frac{y}{\gamma\alpha' - \gamma'\alpha} + \mathrm{C}^2 \frac{z}{\alpha\beta' - \alpha'\beta} = 0,$$

$$\mathrm{A}'^2 \frac{x}{\beta\gamma' - \beta'\gamma} + \mathrm{B}'^2 \frac{y}{\gamma\alpha' - \gamma'\alpha} + \mathrm{C}'^2 \frac{z}{\alpha\beta' - \alpha'\beta} = 0,$$

and for the coordinates of the intersection of these tangents, we have

$$\frac{x}{\beta\gamma' - \beta'\gamma} : \frac{y}{\gamma\alpha' - \gamma'\alpha} : \frac{z}{\alpha\beta' - \alpha'\beta}$$

$$= \mathrm{B}^2\mathrm{C}'^2 - \mathrm{B}'^2\mathrm{C}^2 : \mathrm{C}^2\mathrm{A}'^2 - \mathrm{C}'^2\mathrm{A}^2 : \mathrm{A}^2\mathrm{B}'^2 - \mathrm{A}'^2\mathrm{B}^2.$$

$$\mathrm{BC}' - \mathrm{B}'\mathrm{C} = f\left\{ -f(\beta\gamma' - \beta'\gamma) + g(\gamma\alpha' - \gamma'\alpha) + h(\alpha\beta' - \alpha'\beta)\right\}$$

$$\mathrm{BC}' + \mathrm{B}'\mathrm{C} = 2gh\alpha\alpha' + f\left\{ f(\beta\gamma' + \beta'\gamma) + g(\gamma\alpha' + \gamma'\alpha) + h(\alpha\beta' + \alpha'\beta)\right\}.$$

To satisfy the equation

$$\sqrt{\left\{f(\beta\gamma' - \beta'\gamma)\right\}} + \sqrt{\left\{g(\gamma\alpha' - \gamma'\alpha)\right\}} + \sqrt{\left\{h(\alpha\beta' - \alpha'\beta)\right\}}$$

write $\qquad f = \dfrac{a^2}{\beta\gamma' - \beta'\gamma},\quad g = \dfrac{b^2}{\gamma\alpha' - \gamma'\alpha},\quad h = \dfrac{c^2}{\alpha\beta' - \alpha'\beta},$

and therefore $a + b + c = 0$; we then have

$$-f(\beta\gamma' - \beta'\gamma) + g(\gamma\alpha' - \gamma'\alpha) + h(\alpha\beta' - \alpha'\beta) = -a^2 + b^2 + c^2 = -2bc\,;$$

and thence

$$f\left\{ -f(\beta\gamma' - \beta'\gamma) + g(\gamma\alpha' - \gamma'\alpha) + h(\alpha\beta' - \alpha'\beta)\right\} = \frac{a^2}{\beta\gamma' - \beta'\gamma}(-2bc),$$

and the equations become

$$x : y : z = a(\mathrm{BC}' + \mathrm{B}'\mathrm{C}) : b(\mathrm{CA}' + \mathrm{C}'\mathrm{A}) : c(\mathrm{AB}' + \mathrm{A}'\mathrm{B}),$$

where $\mathrm{BC}' + \mathrm{B}'\mathrm{C}$, $\mathrm{CA}' + \mathrm{C}'\mathrm{A}$, $\mathrm{AB}' + \mathrm{A}'\mathrm{B}$, substituting therein for f, g, h the values $\dfrac{a^2}{\beta\gamma' - \beta'\gamma}$, $\dfrac{b^2}{\gamma\alpha' - \gamma'\alpha}$, $\dfrac{c^2}{\alpha\beta' - \alpha'\beta}$, are respectively functions of the fourth degree in a, b, c; hence (a, b, c) being connected by the relation $a + b + c = 0$, x, y, z are proportional to quintic functions of (a, b, c), or what is the same thing, writing $a, b, c = 1, \theta. -1 - \theta$, then x, y, z are proportional to quintic functions of θ, that is, the locus is a unicursal quintic curve. •

That the curve passes through the points $(\alpha', \beta', \gamma')$ and (α, β, γ) appears by considering the conics $fyz + gzx + hxy = 0$, which pass through these points respectively.

For the first of these conics we have $f : g : h = \alpha(\beta\gamma' - \beta'\gamma) : \beta(\gamma\alpha' - \gamma'\alpha) : \alpha(\beta\gamma' - \beta'\gamma)$; the equation

$$\mathrm{A}^2 \frac{x}{\beta\gamma' - \beta'\gamma} + \mathrm{B}^2 \frac{y}{\gamma\alpha' - \gamma'\alpha} + \mathrm{C}^2 \frac{z}{\alpha\beta' - \alpha'\beta} = 0,$$

reduces itself to $x(\beta\gamma' - \beta'\gamma) + y(\gamma\alpha' - \gamma'\alpha) + z(\alpha\beta' - \alpha'\beta) = 0$, and as the other equation

$$\mathrm{A}'^2 \frac{x}{\beta\gamma' - \beta'\gamma} + \mathrm{B}'^2 \frac{y}{\gamma\alpha' - \gamma'\alpha} + \mathrm{C}'^2 \frac{z}{\gamma\alpha - \gamma'\alpha} = 0,$$

is that of a line through $(\alpha', \beta', \gamma')$ the two lines meet of course in the point $(\alpha', \beta', \gamma')$. And the like for the conic $f : g : h = \alpha'(\beta\gamma' - \beta'\gamma) : \beta'(\gamma\alpha' - \gamma'\alpha) : \gamma'(\alpha\beta' - \alpha'\beta)$.

If the triangle is equilateral, and (x, y, z) are respectively proportional to the perpendicular distances from the three sides, then we have for the circular points at infinity

$$(\alpha, \beta, \gamma) = (1, \omega, \omega^2). \quad (\alpha', \beta', \gamma') = (1, \omega^2, \omega),$$

where ω is an imaginary cube root of unity. These values give

$$\beta\gamma' - \beta'\gamma = \gamma\alpha' - \gamma'\alpha = \alpha\beta' - \alpha'\beta = \omega^2 - \omega$$

$$\alpha\alpha' = \beta\beta' = \gamma\gamma' = 1, \ \beta\gamma' + \beta'\gamma = \gamma\alpha' + \gamma'\alpha = \alpha\beta' + \alpha'\beta = -1 \ ;$$

and the expressions for (x, y, z) take the form

$$\begin{aligned} x : y : z = a & \left\{2b^2c^2 - a^2(a^2 + b^2 + c^2)\right\} \\ : b & \left\{2c^2a^2 - b^2(a^2 + b^2 + c^2)\right\} \\ : c & \left\{2a^2b^2 - c^2(a^2 + b^2 + c^2\right\}, \end{aligned}$$

or, what is the same thing, reducing by means of the relation $a + b + c = 0$, $x : y : z =$

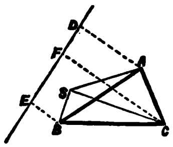
$$a(a^4 - 2a^2bc - 2b^2c^2) : b(b^4 - 2b^2ca - 2c^2a^2) : c(c^4 - 2c^2ab - 2a^2b^2),$$

and the equation of the curve is obtained by eliminating (a, b, c) from these equations and the before mentioned equation $a + b + c = 0$.

N.B.—The above is a particular case of the following general theorem of M. CHASLES:—If the conics of a system (μ, ν) all of them touch the line QQ′, the locus of the intersection of the tangents through Q, Q′ to each conic of the series is a curve of the order $\frac{1}{2}\mu + \nu$, having a $(\frac{1}{2}\mu)$-tuple point at the points Q, Q′ respectively.

II. *Solution by* F. D. THOMSON, M.A.

To find the locus of the foci of a parabola described about a given triangle.



Let DEF be directrix, S the focus of one of the parabolas about ABC. Draw AD, BE, CF perpendicular to DEF; then AS = AD, BS = BE, CS = CF. Let $\lambda x + \mu y + \nu z = 0$ be the *trilinear* equation to DEF; then putting $\Delta = 2$ area of ABC, and $k^2 = \lambda^2 + \mu^2 + \nu^2 - 2\mu\nu \cos A - \&c.$, the perpendicular on DEF from a point $(x', y', z') =$

$$\frac{\lambda x' + \mu y' + \nu z'}{k}, \quad \text{therefore AD} = \frac{\lambda}{k}\frac{\Delta}{a};$$

therefore $$\frac{a \,.\, \text{AS}}{\lambda} = \frac{b \,.\, \text{BS}}{\mu} = \frac{c \,.\, \text{CS}}{\nu} = \frac{\Delta}{k},$$

therefore $a^2 \,.\, \text{AS}^2 + b^2 \,.\, \text{BS}^2 + c^2 \,.\, \text{CS}^2 - 2bc \,.\, \text{BS} \,.\, \text{CS} \cos A - \&c. = \Delta^2 \ldots$ (i).

But if (x, y, z) be the coordinates of S, we have

$$\text{AS}^2 \sin^2 A = y^2 + z^2 + 2yz \cos A, \ \text{BS}^2 \sin^2 B = z^2 + x^2 + 2zx \cos B,$$

$$\text{CS}^2 \sin^2 C = x^2 + y^2 + 2xy \cos C;$$

hence, substituting in (i.) we get the equation to the locus of S.

2263. (Proposed by Hugh Godfray, M.A.)—Two chords are drawn at random in a circle; find the chance that they will intersect, supposing a random chord to mean any line joining two points of the circle, all lengths less than the diameter being equally probable, and two chords drawn at random to mean two such chords inclined at any angle, all inclinations being equally probable.

Solution by J. M. Wilson, M.A., F.G.S.

The probability of one chord having a length between l and $l+\delta l$ is $\frac{\delta\lambda}{2r}$, where r is the radius; also the probability of the second chord being less than the first is $\frac{l}{2r}$; and the probability of the second chord lying between l' and $l'+\delta l'$, less than the first, is $\frac{dl'}{l}$; moreover the probability of chords of lengths l, l' $(l > l')$ intersecting within the circle, if all inclinations are equally probable, is $\frac{2}{\pi}\sin^{-1}\frac{l'}{2r}$; hence the whole probability required is

$$\int_0^{2r}\int_0^{l}\frac{2}{\pi}\sin^{-1}\frac{l'}{2r}\cdot\frac{l}{2r}\cdot\frac{dl}{2r}\cdot\frac{dl'}{l}=\frac{1}{2\pi r^2}\int_0^{2r}\int_0^{l}\sin^{-1}\frac{l'}{2r}\,dl\,dl'.$$

Put $\sin^{-1}\frac{l'}{2r}=\lambda'$, $\sin^{-1}\frac{l}{2r}=\lambda$, and the integral becomes

$$\frac{1}{\pi r}\int_0^{2r}dl\int_0^{\lambda}\lambda\cos\lambda\,d\lambda=\frac{1}{\pi r}\int_0^{2r}(\lambda\sin\lambda+\cos\lambda-1)\,dl$$

$$=\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}(\lambda\sin\lambda\cos\lambda+\cos^2\lambda-\cos\lambda)\,d\lambda$$

$$=\frac{2}{\pi}\left(-\tfrac{1}{4}\lambda\cos 2\lambda+\tfrac{3}{8}\sin 2\lambda+\tfrac{1}{2}\lambda-\sin\lambda\right),\text{ between the limits,}$$

$$=\frac{2}{\pi}\left\{\frac{3\pi}{8}-1\right\}=\frac{3\pi-8}{4\pi}.$$

2268. (Proposed by G. C. De Morgan, M.A.)—Find the values of $n_1, n_2, n_3, \ldots n_k$ which make $\frac{n_1^2+n_2^2+n_3^2+\ldots\ldots}{(a_1n_1+a_2n_2+\ldots\ldots)^2}$ or $\frac{\Sigma n^2}{(\Sigma an)^2}$ a minimum, $n_1, n_2, n_3, \ldots\ldots$ being connected by the equations

$$\Sigma bn=b_1n_1+b_2n_2+\ldots\ldots=0,\quad \Sigma cn=c_1n_1+c_2n_2+\ldots\ldots=0, \text{ \&c. \&c.},$$

the number of these equations being less than $k-1$.

Solution by the PROPOSER.

By equating the differential of $\frac{\Sigma n^2}{(\Sigma an)^2}$ to 0, we get as the condition for a minimum $\{n_1 . \Sigma an - a_1 . \Sigma n^2\} . \delta n_1 + \{n_2 . \Sigma an - a_2 . \Sigma n^2\} . \delta n_2 + \ldots . = 0$, which is to be true for all values of $\delta n_1, \delta n_2, \ldots .$ consistent with the relations

$$b_1 . \delta n_1 + b_2 . \delta n_2 + b_3 \delta n_3 + \ldots . = 0,\ c_1 . \delta n_1 + c_2 . \delta n_2 + c_3 . \delta n_3 + \ldots . = 0, \text{ \&c.}$$

Hence we have

$$n_1 . \Sigma an - a_1 . \Sigma n^2 + \lambda . b_1 + \mu . c_1 + \ldots . = 0,$$
$$n_2 . \Sigma an - a_2 . \Sigma n^2 + \lambda . b_2 + \mu . c_2 + \ldots . = 0, \text{ \&c.}$$

where $\lambda, \mu, \nu \ldots .$ are some quantities undetermined. These may be written

$$n_1 = \text{A} . a_1 + \text{B} . b_1 + \text{C} . c_1 + \ldots .,\ n_2 = \text{A} . a_2 + \text{B} . b_2 + \text{C} . c_2 + \ldots ., \text{ \&c.}$$

Multiplying these by a_1, a_2, and adding, we get

$$\left.\begin{array}{lll} & \Sigma an & = \text{A} . \Sigma a^2 + \text{B} . \Sigma ab + \text{C} . \Sigma ac + \ldots . \\ \text{similarly} & \Sigma bn & = 0 = \text{A} . \Sigma ab + \text{B} . \Sigma b^2 + \text{C} . \Sigma bc + \ldots . \\ \text{and} & \Sigma cn & = 0 = \text{A} . \Sigma ac + \text{B} . \Sigma bc + \text{C} . \Sigma c^2 + \ldots . \\ & & \quad \text{\&c.} \qquad\qquad \text{\&c.} \end{array}\right\} \ldots (1).$$

Hence

$$\Delta . \text{A} = \Sigma an . \begin{vmatrix} \Sigma b^2 & \Sigma bc & \Sigma bd \ldots . \\ \Sigma bc & \Sigma c^2 & \Sigma cd \ldots . \\ \Sigma bd & \Sigma cd & \Sigma d^2 \ldots . \\ \cdot & \cdot & \cdot \quad \cdot \end{vmatrix}$$

$$\Delta . \text{B} = -\Sigma an . \begin{vmatrix} \Sigma ab & \Sigma bc & \Sigma bd \ldots . \\ \Sigma ac & \Sigma c^2 & \Sigma cd \ldots . \\ \Sigma ad & \Sigma cd & \Sigma d^2 \ldots . \\ \cdot & \cdot & \cdot \quad \cdot \end{vmatrix},$$

and so on, Δ being the determinant formed from the coefficients of A, B, C,.... in the equations (1).

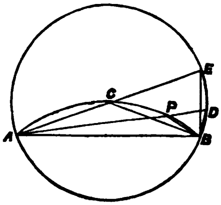
Substituting the values of A, B, C.... obtained from these equations, we get the values of $n_1, n_2, n_3, \ldots .$, which are proportional to

$$\begin{vmatrix} a_1 & b_1 & c_1 \ldots . \\ \Sigma ab & \Sigma b^2 & \Sigma bc \ldots . \\ \Sigma ac & \Sigma bc & \Sigma c^2 \ldots . \\ \cdot & \cdot & \cdot \quad \cdot \end{vmatrix},\ \begin{vmatrix} a_2 & b_2 & c_2 \ldots . \\ \Sigma ab & \Sigma b^2 & \Sigma bc \ldots . \\ \Sigma ac & \Sigma bc & \Sigma c^2 \ldots . \\ & \cdot & \cdot \quad \cdot \end{vmatrix}, \text{ \&c.}$$

The determination of the values of $n_1, n_2 \ldots .$ which make $\frac{\Sigma n^2}{(\Sigma an)^2}$ a minimum under the conditions given above is necessary in a proof of the *method of minimum squares* given by Mr. AIRY in his *Theory of Errors of Observations.* He does not solve the resulting equations analytically, but shows that they are satisfied by certain values of $n_1, n_2, \ldots .$. See page 76 of the work referred to.

2112. (Proposed by C. M. INGLEBY, LL.D.)—To find a point P in a given circular arc ABC, so that the chords AP, BP drawn therefrom to the ends of the chord AB may be given or a maximum.

Solution by J. Dale; H. Tomlinson; W. Chadwick; H. Murphy; W. H. Laverty; *the* Proposer; *and others.*



With C the middle point of the given arc as centre, and CA or CB as radius, draw a circle; and from either end of AB, draw in this circle a chord AD equal to the given line and cutting the given arc in P; join BP, then the chords AP, PB are together equal to AD. For join AC and produce it to meet the circle in E; and join EB, DB; then the triangles BCE, BPD are evidently isosceles. The sum of AP and PB is a *minimum* when the given line is equal to AB; and a *maximum* when the given line is equal to AC + BC, this is, when P coincides with C.

2253. (Proposed by W. K. Clifford.)—If four circles have a common radical centre, it is possible to find four planes which intersect, two and two, at angles equal to those at which the circles intersect, but not otherwise.

Solution by the Rev. R. Townsend, F.R.S.

The angles of intersection, two and two, of any four circles in a plane being the same as those of the four spheres having the same centres and radii; and the angles of intersection, two and two, of four spheres being equal to those of four planes only when they pass through a common point (as is evident from the theory of inversion); therefore, &c.

2265. (Proposed by H. R. Greer, B.A.)—A variable point determines with reference to a fixed cubic curve two covarying conics, viz., one passing through the six points of contact of tangents drawn from the point, and another passing through the six points where these tangents meet the curve again: prove that these two conics have always double contact with one another. The chord of contact is the line-polar of the point with reference to the cubic.

Solution by the Rev. R. Townsend, F.R.S.

Denoting by (x', y', z') the coordinates of the variable point, to any triangle of reference, by (x, y, z) those of any point on the cubic, by $u = 0$ the equation of the curve, and by Δ the symbol of operation $\left(x'\frac{d}{dx} + y'\frac{d}{dy} + z'\frac{d}{dz}\right)$; then, the condition (obtained by substituting $x+\lambda x'$, $y+\lambda y'$, $z+\lambda z'$ for x, y, z in u, and equating to 0 the discriminant of the resulting quadratic in λ) that the line connecting the two points (x, y, z) and (x', y', z') should meet the curve again at two coincident points, being

$$16 . (\Delta u) . (\Delta^3 u) = 6 . (\Delta^2 u)^2,$$

therefore, &c.; the latter equation representing a conic having double contact with the conic $\Delta u = 0$ at its two points of intersection, real or imaginary, with the line $\Delta^2 u = 0$.

N.B.—The corresponding property in geometry of three dimensions appears evidently in precisely the same manner.

2242. (Proposed by W. S. BURNSIDE, M.A.)—If a surface of the second order circumscribe a tetrahedron, and cut another in two plane sections, and if one of these sections pass through a fixed point; find the envelope of the other.

Solution by the REV. R. TOWNSEND, F.R.S.

If $S = ax^2 + by^2 + cz^2 + dw^2 + \&c. = 0$, be the equation of the fixed quadric referred to the four faces of the tetrahedron, $L = px + qy + rz + sw = 0$ and $L' = p'x + q'y + r'z + s'w = 0$ those of its two planes of intersection with the variable quadric $S - LL' = 0$, and $(\alpha, \beta, \gamma, \delta)$ the coordinates of the fixed point; then, since by hypothesis the quadric $S - LL' = 0$ passes through the four vertices of the tetrahedron of reference, and since consequently $pp' = a$, $qq' = b$, $rr' = c$, $ss' = d$, therefore, if the plane $L = px + qy + rz + sw = 0$ turn round the point $\alpha\beta\gamma\delta$, the plane $L' = \frac{a}{p}x + \frac{b}{q}y + \frac{c}{r}z + \frac{d}{s}w = 0$, in which the parameters p, q, r, s are connected by the relation $\alpha p + \beta q + \gamma r + \delta s = 0$, envelopes the quartic $\sqrt{(a\alpha x)} + \sqrt{(b\beta y)} + \sqrt{(c\gamma z)} + \sqrt{(d\delta w)} = 0$, which intersects the four faces of the tetrahedron each in a double conic touching its three lines of intersection with the other three.

N.B.—That the four faces of the tetrahedron are particular positions of the variable plane L', is evident *a priori* from the consideration that when the variable plane L passes through one of its four vertices, the variable quadric passing through the four, then necessarily breaks up into that plane and the opposite face; as appears also from the equations of the two planes, three of the four parameters p, q, r, s then becoming infinite.

2251. (Proposed by T. SAVAGE, M.A.)—From an urn containing n counters, numbered 1, 2, 3,......n, k drawings are made, the counter drawn being replaced each time. If the highest number drawn be m, prove that the most probable value of n is the integer next above, or next below,

$$\sqrt[k]{2} . (m-1).$$

Solution by SAMUEL ROBERTS, M.A.

The most probable circumstances, under which m would, under the conditions stated, be the highest number drawn, are those which make the chances of failure of the event by defect and excess equal. There are $(m-1)^k$ possible cases of failure by defect, and n^k-m^k possible cases of failure by excess, and the most probable state of things is

$$n^k = (m-1)^k + m^k.$$

Now, if $n^k = 2(m-1)^k$, the value of n is too little, and if $n^k = \left\{\sqrt[k]{2}.(m-1)+1\right\}^k$, the value of n is too great, as will be seen by comparison. Consequently the integer value of n which most nearly satisfies the foregoing condition must be that given in the question.

2300. (Proposed by S. ROBERTS, M.A.)—Show that 24 cusped cubic curves can usually be drawn through 7 given points.

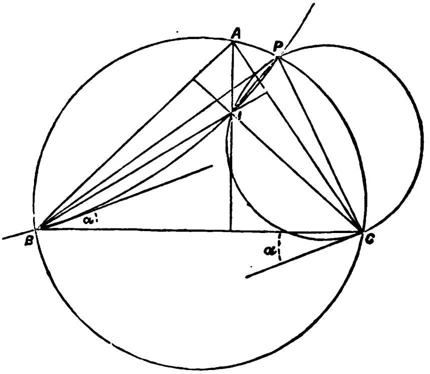
Solution by FRANCIS A. TARLETON, M.A.

The Hessian of the Hessian of a cubic curve $U = 0$ is known to be of the form $4S^2U-TH$, where H is the Hessian of U, and S and T are functions of the coefficients of U of the 4th and 6th degree. (SALMON'S *Higher Plane Curves*, p. 184.) Now if $U = 0$ have a cusp, the tangent at which is $x = 0$, H is of the form x^2y, and therefore the Hessian of H vanishes identically; hence, in this case, $4S^2U-TH$ must vanish identically: but if U and H be referred to x and y as coordinates, the lowest term in U is ax^2, which cannot be destroyed by any term in H, as the lowest term there is x^2y; hence, $S = 0$, and therefore $T = 0$; hence $S = 0$, $T = 0$ are the conditions that U should have a cusp.

Now if U pass through seven given points, it will be of the form $lU_1+mU_2+nU_3$, where U_1, U_2, U_3 are three fixed cubics through the seven points: if then l, m, n be determined by the equations $S = 0$ and $T = 0$, $U = 0$ will be a cubic passing through the seven given points and having a cusp. Now $S = 0$, $T = 0$ being of the 4th and 6th degree in the coefficients of U give 24 values for the ratios $m : l$, $n : l$; hence there are 24 cubics fulfilling the required conditions.

2272. (Proposed by Professor HIRST.)—Through each angle of a triangle, and through the intersection of its three perpendiculars, a circle is drawn. Prove that if the tangents to the three circles at the angles of the triangle be parallel to one another, the circles will intersect in a second point. Find the locus of the latter point when the common direction of the three tangents varies.

I. *Solution by* S. WATSON; Rev. R. TOWNSEND, F.R.S.; F. D. THOMSON, M.A.; H. TOMLINSON; *and others.*



Let I be the intersection of the perpendiculars; P the intersection of the circles through (B, I) and (C, I); and α the inclination of the tangents at B and C to those circles with BC. Then

$$\angle BPC = BPI + CPI$$
$$= IBC - \alpha + ICB + \alpha$$
$$= 90^\circ - C + 90^\circ - B$$
$$= A.$$

Hence P is a point in the circle about ABC. In like manner the circles through (B, I) and (A, I) meet in a point on the same circle about ABC; consequently this must be P, and the three circles all pass through P; also the locus of P, when α varies, is the circle about ABC.

II. *Solution by* T. SAVAGE, M.A.; *and* H. R. GREER, B.A.

1. The theorem stated in the first part is comprehended in the following more general theorem:—From the angles of a triangle let there be drawn three lines parallel to one another A∞, B∞, C∞; from any point O let there be drawn three lines Ox, Oy, Oz, parallel to the three sides BC, CA, AB, respectively; let A∞ meet Ox in α, B∞ meet Oy in β, C∞ meet Oz in γ; then the six points A, B, C, α, β, γ lie on a conic. This may be readily proved geometrically.

2. Consequently, the middle points of Aα, Bβ, Cγ lie on a straight line; these are the centres of the circles in Dr. HIRST'S question, viz., when O is the intersection of the perpendiculars of the triangle. Call this the line L.

3. The envelope of L is a conic touching the sides of the triangle ABC.

4. Using trilinear coordinates, ABC being the triangle of reference; let the coordinates of O be α, β, γ; then the equation of this envelope is

$$a^2 D^2 x^2 + b^2 E^2 y^2 + c^2 F^2 y^2 - 2\,(ab\mathrm{DE}xy + bc\mathrm{EF}yz + ca\mathrm{FD}zx) = 0,$$

where $\quad D = b\beta + c\gamma, \quad E = c\gamma + a\alpha, \quad F = a\alpha + b\beta,$

and, of course, a, b, c mean the lengths of the sides of ABC. The envelope is independent of the direction of A∞.

5. In Dr. HIRST'S question, D, E, F become, simply, $a \cos A$, $b \cos B$, $c \cos C$; hence, from the second part of the question, we conclude that the above equation, these values being understood, is that of the inscribed conic whose foci are the intersection of the perpendiculars and the centre of the circumscribing circle.

6. The centre of this conic is the centre of the nine-point circle; its major axis is equal to the radius of the circumscribing circle.

2118. (Proposed by the Rev. A. F. Torry, M.A.)—Find the ratios $l : m : n$ from the equations

$$l^2 \cos 2\alpha + m^2 \cos 2\beta + n^2 \cos 2\gamma$$
$$+ \kappa \left\{ mn \cos(\beta+\gamma) + nl \cos(\gamma+\alpha) + lm \cos(\alpha+\beta) \right\} = 0,$$
$$l^2 \sin 2\alpha + m^2 \sin 2\beta + n^2 \sin 2\gamma$$
$$+ \kappa \left\{ mn \sin(\beta+\gamma) + nl \sin(\gamma+\alpha) + lm \sin(\alpha+\beta) \right\} = 0,$$

where $\kappa = \pm 2$ or -1.

Solution by W. H. Laverty; S. W. Bromfield; H. Tomlinson; *and others.*

If we multiply the second of these two equations by $\sqrt{(-1)}$ or i, and then alternately add and subtract them, we shall get

$$l^2\epsilon^{2\alpha i} + m^2\epsilon^{2\beta i} + n^2\epsilon^{2\gamma i} + \kappa\left\{ mn\epsilon^{(\beta+\gamma)i} + nl\epsilon^{(\gamma+\alpha)i} + lm\epsilon^{(\alpha+\beta)i} \right\} = 0 \ldots (1),$$

$$l^2\epsilon^{-2\alpha i} + m^2\epsilon^{-2\beta i} + n^2\epsilon^{-2\gamma i} + \kappa\left\{ mn\epsilon^{-(\beta+\gamma)i} + nl\epsilon^{-(\gamma+\alpha)i} + lm\epsilon^{-(\alpha+\beta)i} \right\} = 0 \ldots (2).$$

If $\kappa = +2$ we have $l\epsilon^{\alpha i} + m\epsilon^{\beta i} + n\epsilon^{\gamma i} = 0$, and $l\epsilon^{-\alpha i} + m\epsilon^{-\beta i} + n\epsilon^{-\gamma i} = 0$;
whence $l : m : n = \sin(\beta-\gamma) : \sin(\gamma-\alpha) : \sin(\alpha-\beta)$.
If $\kappa = -2$ we have, by reduction,

$$l^{\frac{1}{2}}\epsilon^{\frac{1}{2}\alpha i} + m^{\frac{1}{2}}\epsilon^{\frac{1}{2}\beta i} + n^{\frac{1}{2}}\epsilon^{\frac{1}{2}\gamma i} = 0, \text{ and } l^{\frac{1}{2}}\epsilon^{-\frac{1}{2}\alpha i} + m^{\frac{1}{2}}\epsilon^{-\frac{1}{2}\beta i} + n^{\frac{1}{2}}\epsilon^{-\frac{1}{2}\gamma i} = 0;$$

whence $l : m : n = \sin^2 \frac{1}{2}(\beta-\gamma) : \sin^2 \frac{1}{2}(\gamma-\alpha) : \sin^2 \frac{1}{2}(\alpha-\beta)$.
If $\kappa = -1$; let $1, \omega, \omega^2$ be the cube roots of 1, so that $\omega = -\frac{1}{2}(1 + i\sqrt{3})$; then solving (1) and (2) as quadratics in $l\epsilon^{\alpha i}$ and $l\epsilon^{-\alpha i}$ we shall get

$$l\epsilon^{\alpha i} + \omega m\epsilon^{\beta i} + \omega^2 n\epsilon^{\gamma i} = 0, \text{ and } l\epsilon^{-\alpha i} + \omega m\epsilon^{-\beta i} + \omega^2 n\epsilon^{-\gamma i} = 0;$$

whence $l : m : n = \omega^2 \sin(\beta-\gamma) : \omega \sin(\gamma-\alpha) : \sin(\alpha-\beta)$.

2250. (Proposed by Professor Cayley.)—From the focal equation $x^2 + y^2 = (lx+n)^2$, of a conic, deduce the remaining three focal equations.

Solution by the Proposer.

We are to find α, β, L, M, N such that the equation

$$(x-\alpha)^2 + (y-\beta)^2 = (\mathrm{L}x + \mathrm{M}y + \mathrm{N})^2,$$

may be identical with the given equation. It is at once seen that we must have $\mathrm{M} = 0$ or else $\mathrm{L} = 0$; the first supposition gives two solutions, one of which is the given equation itself, the other is

$$\left(x - \frac{2ln}{1-l^2}\right)^2 + y^2 = \left\{ lx - n\frac{1+l^2}{1-l^2} \right\}^2.$$

The second supposition, L = 0, gives two solutions, which only differ by the sign of $i\,(=\sqrt{-1})$, viz. these are

$$\left(x-\frac{ln}{1-l^2}\right)^2+\left(y\mp\frac{lni}{1-l^2}\right)^2=\frac{-(ly\pm ni)^2}{1-l^2}.$$

There is, of course, no difficulty in *verifying* the identity of each of the three forms with the given form $x^2+y^2=(lx+n)^2$.

1964. (Proposed by the Rev. R. H. Wright, M.A.)—An iron shell, whose outside and inside diameters were respectively 9·8 and 7 inches, was filled with gunpowder, and, being discharged at an elevation of 45°, reached its greatest elevation just over a house half a mile distant, where it burst: find the weight of powder required to discharge it from the mortar, and the height above the house at which the shell burst (supposing the mean weight of 30 cubic inches of gunpowder to be 1 lb.).

Solution by the Proposer; H. Murphy; H. Tomlinson; J. Dale; E. McCormick; *and others.*

The equation to the parabolic path is

$$y=x\tan A-\frac{gx^2}{2V^2\cos^2 A}\;\ldots\ldots\ldots\ldots\ldots\ldots(1);$$

and since the greatest height $=\dfrac{V^2}{2g}\sin^2 A$,

substituting this value for y in (1), we get

$$V^4-4gV^2+4g^2x^2=0,\ (\text{since } A=45^\circ \text{ by hypothesis});$$

also, by the question, when the shell is at its greatest height, $x=2640$ feet, and since $g=32\cdot2$, we have $V^2=170016$.

Now the weight of a shell whose external and internal diameters are respectively 9·8 and 7 inches, is found by well known rules to be 84·941 lbs., and the powder that can be placed in it will be $\dfrac{7^3}{57\cdot3}=5\tfrac{9}{10}$ lbs. nearly, therefore the whole weight of the charged shell $=84\tfrac{9}{10}+5\tfrac{9}{10}=90\cdot8$ nearly.

Also $V=1600\sqrt{\left(\dfrac{3p}{w}\right)}$, where p = weight of powder and $w=90\cdot8$;

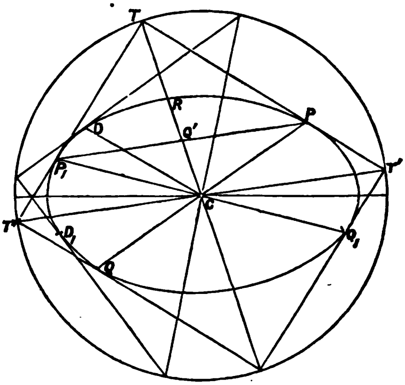
therefore $170016=2560000\left(\dfrac{3p}{90\cdot8}\right)$, whence $p=2$ lbs., nearly.

Again, the greatest height $=\dfrac{V^2}{2g}\cdot\sin^2 A=\dfrac{170016}{128\cdot8}=1320$ feet.

Hence 2 lbs. of gunpowder will carry the shell to the point in the path 1320 feet (or a quarter of a mile) above the house half a mile distant, where, from some uncalculated cause, it is supposed to burst.

Note on Question 1968. By R. Tucker, M.A.

By means of this neat property (*Reprint*, Vol. V., p. 104), the following problem can readily be solved :—Given a pair of semi-conjugate diameters of an ellipse, to find any number of points on the curve.



Let CP (a') and CD (b') be the given semi-conjugate diameters; then with centre C and radius $= \sqrt{(a'^2 + b'^2)}$ draw a circle; through P draw a line parallel to CD meeting the circle in T, T′. Draw the diameter T′CT″, and join TT″; then, by Quest. 1968, the intersection of PP_1, parallel to T′T″, with TT″, gives a point P_1 on the curve.

Then the semi-conjugate (to CP_1) CP_2 can be drawn, and P_2 another point on the curve is found, and so on.

Similarly, starting from CD, we can find D_1, and other points on the ellipse. Moreover points R, R_1, &c. can be obtained by taking CR, &c. harmonic means between CQ′ and CT, &c. And, of course, the opposites to these on the curve are at once obtainable.

2305. (Proposed by W. B. Davis, B.A.)—Compare the greatest and least inscribed squares in a scalene triangle.

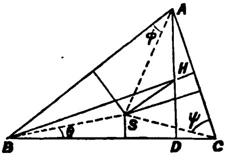
Solution by R. Tucker, M.A.; Rev. J. L. Kitchin, M.A.; H. Tomlinson; E. McCormick; *and others.*

Let p, q be the perpendiculars on the greatest (a) and least (c) sides of the triangle; then (McDowell's *Exercises*, Props. 65, 70), if A, A′ are the greatest and least squares respectively, we have

$$\frac{A'}{A} = \left(\frac{c+q}{cq}\cdot\frac{ap}{a+p}\right)^2 = \left(\frac{c+q}{a+p}\right)^2 = \left(\frac{c+b\sin A}{a+b\sin C}\right)^2.$$

1755. (Proposed by R. Tucker, M.A.)—Conics of the same species are described, passing through the angular points of a triangle, and having one focus coinciding with the point of intersection of the perpendiculars from the angles on the opposite side; show that the locus of the other focus is a conic.

Solution by the PROPOSER.



Let $S(\alpha, \beta, \gamma)$ be one position of the focus; $\angle SBC=\theta$, $SAB=\phi$, $SCA=\psi$; then $SA = \gamma \operatorname{cosec} \phi$, $SB = \alpha \operatorname{cosec} \theta$, $SC = \beta \operatorname{cosec} \psi$, $AH = a \cot A$, $BH = b \cot B$, $CH = c \cot C$; and we have, by the hypothesis,

$$a \cot A \pm \gamma \operatorname{cosec} \phi = b \cot B \pm \alpha \operatorname{cosec} \theta$$
$$= c \cot C \pm \beta \operatorname{cosec} \psi:$$

also $$\frac{SB}{SC} = \frac{\alpha \sin \psi}{\beta \sin \theta} = \frac{\sin (C-\psi)}{\sin \theta}, \text{ whence } \cot \psi = \frac{\alpha + \beta \cos C}{\beta \sin C};$$

we have, therefore, for determining the locus,

$$\pm \operatorname{cosec} C\,(\alpha^2 + \beta^2 + 2\alpha\beta \cos C)^{\frac{1}{2}} + c \cot C$$
$$= b \cot B \pm \operatorname{cosec} B\,(\alpha^2 + \gamma^2 + 2\alpha\gamma \cos B)^{\frac{1}{2}};$$

arranging and squaring, we get, if $\lambda = b(\cos B - \cos C)$,

$$(b^2 - c^2)\,\alpha^2 + b^2\beta^2 - c^2\gamma^2 + 2b^2\alpha\beta \cos C - 2c^2\alpha\gamma \cos B - \lambda^2 c^2$$
$$= \pm 2\lambda c^2 (\alpha^2 + \gamma^2 + 2\alpha\gamma \cos B)^{\frac{1}{2}},$$

also $$a^2\alpha^2 + b^2\beta^2 - c^2\gamma^2 + 2ab\alpha\beta = 4\Delta^2 - 4\Delta c\gamma;$$

hence $$\alpha^2 (2ac \cos B) + 2bc\alpha\beta \cos B + 2c^2\alpha\gamma \cos B + \lambda^2 c^2 - 4\Delta^2 + 4\Delta c\gamma$$
$$= \mp 2\lambda c^2 (\alpha^2 + \gamma^2 + 2\alpha\gamma \cos B)^{\frac{1}{2}},$$

or $$4\Delta\, c\alpha \cos B + \lambda^2 c^2 - 4\Delta^2 + 4\Delta c\gamma = \mp 2\lambda c^2 (\alpha^2 + \gamma^2 + 2\alpha\gamma \cos B)^{\frac{1}{2}},$$

which when cleared of the radical will give a conic.

Or the question may be solved readily by taking as rectangular axes BC and a perpendicular through B. Let $BD=q$, $AD=p$; then

$$SA = \left\{(q-x)^2 + (p-y)^2\right\}^{\frac{1}{2}}, \quad SC = \left\{(a-x)^2 + y^2\right\}^{\frac{1}{2}};$$

$$\therefore \quad \left\{(q-x)^2 + (p-y)^2\right\}^{\frac{1}{2}} \pm a \cot A = \left\{(a-x)^2 + y^2\right\}^{\frac{1}{2}} \pm c \cot C,$$

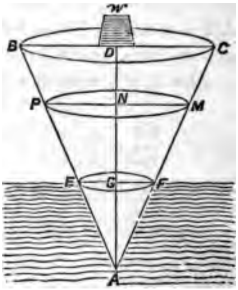
which gives, if $\mu = \pm(c \cot C - a \cot A)$,

$$c^2 - \mu^2 - a^2 + 2(a+q)\,x - 2py = 2\mu \left\{(a-x)^2 + y^2\right\}^{\frac{1}{2}},$$

the equation to a conic, as before.

2225. (Proposed by the Rev. R. H. WRIGHT, M.A.)—A cone floats in a fluid, the densities of both being given; but a weight w being placed on the top of the cone, its draught in the fluid is n times its former draught; show that the weight (w') of the cone is $w' = \dfrac{w}{n^3-1}$.

Solution by H. Murphy; H. Tomlinson; J. Dale; *the* Proposer; E. McCormick; *and many others.*



Let AD be the axis of the cone, EGF the plane of flotation at first, and PNM that after the weight w has been added; then, since the weight in both cases is equal to the weight of the fluid displaced, and this latter varies directly as the cube of the depth, we have

$$w+w' : w' = AN^3 : AG^3 = n^3 : 1,$$

whence we obtain $w' = \dfrac{w}{n^3-1}$.

1746. (Proposed by P. W. Flood.)—Find a square number such that the sum of its digits diminished by 1, and their product increased by 1, shall both be square numbers.

Solution by S. Bills; *the* Proposer; *and others.*

Let x and y denote the digits composing the required number; then we must have

$$10x+y = \square\ldots(1),\quad x+y-1 = \square\ldots(2),\quad xy+1 = \square\ldots(3).$$

Take $x = y+2$, then (3) becomes a square $= (y+1)^2$, and (1) and (2) will become respectively,

$$11y+20 = \square\ldots(4),\quad 2y+1 = \square\ldots(5).$$

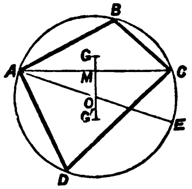
Assume $y = 2p^2+2p$, then (5) will be a square $= (2p+1)^2$, and (4) will become

$$22p^2+22p+20 = \square\ldots(6).$$

Now, (6) will evidently be a square when $p = 1$; this gives $y = 4$ and $x = 6$; and the required number will be 64, the only one which satisfies the conditions of the question. If fractional values were admissible, (6) would admit of an indefinite number of solutions, which may be obtained by well-known methods.

2130. (Proposed by the Editor.)—Find the average area of all the quadrilaterals which can be inscribed in a given circle.

Solution by Stephen Watson.



Let ABCD be a quadrilateral inscribed in a circle whose centre is O, and of which AOE is a diameter. Draw OM perpendicular to the chord AC, passing through its middle point M; and produce OM both ways to G and G′, the centroids (or centres of gravity) of the arcs ABC, ADC. Put OA $= r$, and $\angle$ CAE $= \phi$; then arc CE $= 2r\phi$, and

$$\text{OG} = \frac{r.(\text{chord AC})}{\text{arc ABC}} = \frac{2r\cos\phi}{\pi-2\phi};$$

hence the average area of the triangle ABC, while B takes every position on the arc ABC, is

$$\text{AM}\,.\,\text{MG} = \frac{2r^2\cos^2\phi}{\pi-2\phi} - r^2\sin\phi\cos\phi \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

In like manner the average of the triangle ADC, while D takes every position on the arc ADC, is

$$\frac{2r^2\cos^2\phi}{\pi+2\phi} + r^2\sin\phi\cos\phi \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Multiply (1) and (2) each by $(\pi-2\phi)(\pi+2\phi)\,r^2$, and add; the result, viz.,

$$4\pi r^4\cos^2\phi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

expresses the sum of the areas of the quadrilateral, when B and D take every position on the arcs ABC, ADC. Also, since the points B, C, D may be interchanged in six different ways, and AC may lie on either side of AE, the expression (3) must be multiplied by 12; hence

$$48\pi r^4\int_0^{r\pi}\cos^2\phi\ d\,(\text{arc CE}) = 96\pi r^5\int_0^{\frac{1}{2}\pi}\cos^2\phi\,d\phi = 24\pi^2 r^5$$

expresses the sum of the areas when B, C, D take every possible position on the circumference; and the measure of the number of positions is $8\pi^3 r^3$; therefore the required average is

$$\frac{24\pi^2 r^5}{8\pi^3 r^3} = \frac{3r^2}{\pi} = \frac{3}{\pi^2}\,.\,(\text{area of given circle}).$$

1988. (Proposed by M. Jenkins, B.A.)—If N be a composite number, find the relation which must exist between the periods of the prime factors of N in order that the repetend of the recurring decimal which is equal to $\frac{1}{\text{N}}$ may be complementary.

[By the period of a number is meant the number of digits in the repetend of the recurring decimal which is equal to the reciprocal of the given number, and by a complementary repetend one which has the digits of its last half respectively complementary to 9 with the digits of the first half.]

Solution by the Proposer.

The periods of the prime factors of N must be all even (which, by a theorem, mentioned hereafter, is equivalent to saying that the corresponding repetends must be all complementary) and the periods must also contain the same power of 2.

Examples:	No.	Period.
	7	$\overline{6}$
	23	$\overline{22}$
	161 or 7 × 23	$\overline{66}$ (=L. C. M. of 6 and 22.)
	29	$\overline{28}$
	203 or 7 × 29	84
	89	$\overline{44}$
	5429 or 89 × 61	$\overline{660}$

A stroke above the period signifies that the corresponding repetend is complementary.

Let $N = a^p\, b^q\, c^r \ldots$ where $a, b, c \ldots$ are prime numbers other than 2 or 5, which may be rejected as having no effect on the period, and let $\alpha, \beta, \gamma \ldots$ be the periods of $a, b, c \ldots$ respectively, where $\alpha, \beta, \gamma \ldots$ are certain measures of $a-1, b-1, \ldots$ respectively. Then it may be easily proved that αa^{p-1}, βb^{q-1}, $\gamma c^{r-1} \ldots$ are the periods of a^p, $b^q \ldots$, &c. respectively (with a slight modification if a or $b \ldots = 3$, but this does not affect the proof), and that $La^{p-1}\, b^{q-1}\, c^{r-1} \ldots$ is the period of N where L is the L. C. M. of $\alpha, \beta, \gamma \ldots$

Now since this period is complementary it is even, and therefore since a, b, c are odd, L is even, and since $La^{p-1}b^{q-1} \ldots$ is the period of N, therefore, by a known property, N is a measure of $(10^{\frac{1}{2}La^{p-1}b^{q-1}\ldots} + 1)$, and therefore a, or b, &c., are measures of $(10^{\frac{1}{2}La^{p-1}b^{q-1}\ldots} + 1)$, and are therefore not contained in $(10^{\frac{1}{2}La^{p-1}b^{q-1}\ldots} - 1)$. But if $\frac{1}{2}$L contain α, or $\beta \ldots$ $(10^{\frac{1}{2}La^{p-1}b^{q-1}\ldots} - 1)$ would contain $(10^{\alpha} - 1)$, or $(10^{\beta} - 1)$, and therefore would contain a, or $b \ldots$, which are measures of $(10^{\alpha} - 1)$, $(10^{\beta} - 1) \ldots$, therefore $\frac{1}{2}$L must not contain α, or β, &c., and therefore since L is L. C. M. of $\alpha, \beta, \ldots$, that $\alpha, \beta, \ldots$ must contain same power of 2, is a condition which is *necessary*.

The condition is also sufficient, for if $\alpha, \beta \ldots$ contain the same power of 2, L contains the same power of 2, therefore $\frac{1}{2}$L does not contain α, therefore. $10^{\frac{1}{2}La^{p-1}b^{q-1}\ldots} - 1$ does not contain a, and therefore, since $(10^{La^{p-1}b^{q-1}\ldots} - 1)$ contains a^p, $(10^{\frac{1}{2}La^{p-1}b^{q-1}} + 1)$ contains a^p. In like manner $(10^{\frac{1}{2}La^{p-1}b^{q-1}\ldots} + 1)$ contains $b^q, \ldots$ and therefore contains N, and that N should be a measure of a number of the form $(10^x + 1)$ is known to be a necessary and sufficient condition that the repetend of N should be complementary.

If N be a prime, or the power of a single prime, then, if its period be even, the repetend will be complementary, as may easily be proved.

1991. (Proposed by Professor CAYLEY.)—Given a point and three lines; it is required to draw through the point a plane meeting the three lines in three points equidistant from the given point.

Solution by the PROPOSER.

Let O be the given point, $OA' = a$, $OB' = b$, $OC' = c$ the perpendiculars let fall from O on the given lines respectively. Take θ an arbitrary line, and from the points A′, B′, C′ measure off on the three lines respectively the distances $A'A = \pm\surd(\theta^2-a^2)$, $B'B = \pm\surd(\theta^2-b^2)$, $C'C = \pm\surd(\theta^2-c^2)$, or, considering each radical as containing implicitly the sign $\pm$, what is the same thing, the distances $A'A = \surd(\theta^2-a^2)$, $B'B = \surd(\theta^2-b^2)$, $C'C = \surd(\theta^2-c^2)$, then we have $OA = OB = OC\,(=\theta)$; and consequently the problem is to determine θ in such wise that the plane ABC may pass through the given point O: for we shall then have through O a plane meeting the three given lines in the points A, B, C equidistant from O.

The coordinates of A, B, C are linear functions of the radicals $\surd(\theta^2-a^2)$, $\surd(\theta^2-b^2)$, $\surd(\theta^2-c^2)$ respectively. Taking O as origin, the condition in order that the plane ABC may pass through O is

$$\begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix} = 0,$$

and substituting for the coordinates their values in terms of θ, this is an equation linear in each of the three radicals, or say, an equation of the form

$$\left\{\surd(\theta^2-a^2),\,1\right\}\left\{\surd(\theta^2-b^2),\,1\right\}\left\{\surd(\theta^2-c^2),\,1\right\} = 0.$$

But we may represent any one of the three radicals, say $\surd(\theta^2-c^2)$ by a single letter s; and this being so, we have $\surd(\theta^2-a^2) = \surd(s^2+c^2-a^2) = \surd P$ suppose, and $\surd(\theta^2-b^2) = \surd(s^2+c^2-b^2) = \surd Q$ suppose; and it is to be observed that there is no loss of generality in assuming that the distance $C'c = s$ is measured off from C′ in a determinate sense, for as s passes from $-\infty$ to $+\infty$, we thus obtain for c every position whatever on the line in question; whereas the other two distances A′A, B′B, represented by the radicals $\surd P$ and $\surd Q$ respectively, remain each of them with the double sense $\pm$. The equation in s is of the form

$$(s,1)(\surd P,1)(\surd Q,1) = 0,$$

or, what is the same thing, it is of the form

$$\alpha\surd(PQ) + \beta\surd P + \gamma\surd Q + \delta = 0,$$

where $(\alpha, \beta, \gamma, \delta)$ are respectively linear functions of s.

Proceeding to rationalise the equation, we have first

$$\alpha^2PQ + 2\alpha\delta\sqrt{(PQ)} + \delta^2 = \beta^2P + \gamma^2Q + 2\beta\gamma\sqrt{(PQ)},$$

and then finally $(\alpha^2PQ - \beta^2P - \gamma^2Q + \delta^2)^2 = 4(\beta\gamma - \alpha\delta)^2 PQ$, which, observing that P, Q are each of them of the second order in s, is an equation of the twelfth order in s; that is, the number of solutions is = 12.

The solution of the problem is greatly simplified when $a = b = c$, that is, when the three given lines are tangents to a sphere having its centre at the given point. We have in this case $\sqrt{P} = \pm s$, $\sqrt{Q} = \pm s$, or the equation in s is

$$(s, 1)(\pm s, 1)(\pm s, \overline{1}) = 0;$$

that is, the equation of the twelfth order breaks up into four equations each of the third order. The geometrical theory may also be further developed. In fact, assuming on each of the three lines respectively a certain sense as positive (and thus isolating a set of three solutions) the construction is, on the three lines, from the points A′, B′, C′ respectively, measure off the distances $A'A = B'B = C'C = s$. Then the points A, B, C form on the three lines respectively three homographic series; that is, the lines BC, CA, AB are respectively generating lines of three hyperboloids, viz., hyperboloids which pass respectively through the second and third lines, the third and first lines, and the first and second lines. Taking the given point O as the centre of projection, and projecting the whole figure on any plane whatever, the projections of the lines BC are the tangents of a conic which is the projection of the visible contour of the hyperboloid generated by the lines BC; and the like for the lines CA and AB. Hence in the projection, or plane figure, we have a triangle whereof the sides A′, B′, C′ are the projections of the three given lines respectively; inscribed in this triangle we have a variable triangle ABC, such that the side

BC envelopes a conic, say (A), which touches B′ and C′,
CA envelopes a conic, say (B), which touches C′ and A′,
AB envelopes a conic, say (C), which touches A′ and B′.

The conics (A) (B) (C) have three common tangents, say L, M, N; the conics

(B) and (C) having besides the common tangent A′,
(C) and (A) having besides the common tangent B′,
(A) and (B) having besides the common tangent C′,

so that the common tangents of the conics (B) and (C), (C) and (A), (A) and (B) are the lines A′, B′, C′ each once, and the lines L, M, N each three times. In the entire series of triangles ABC there are three triangles which degenerate into the lines L, M, N respectively, these being in fact the projections of the triangles ABC of the solid figure which lie in a plane with O. Or, what is the same thing, the planes of the required triangles ABC of the solid figure are the planes through O and the three lines L, M, and N, respectively.

1993. (Proposed by T. Cotterill, M.A.)—If P is a point on a circle, in which A and B are fixed points on a diameter at equal distances from its centre, the curve envelope of lines cutting harmonically the two circles whose centres are A and B and radii AP, BP respectively, is independent of the position of P on the circle.

I. *Solution by* ARCHER STANLEY.

Let c be the centre of the circle upon which P is situated, and L any right line cut harmonically in a, a' and b, b' by two circles (A) and (B) whose radii are AP and BP, respectively. A perpendicular let fall from A upon L will bisect aa' in M, and by a well known theorem of harmonic section we shall have

$$Ma^2 = Mb \,.\, Mb'.$$

But $\quad Ma^2 = Aa^2 - AM^2 = AP^2 - AM^2$, and $Mb \,.\, Mb' = BM^2 - BP^2$,

hence $\quad BM^2 - BP^2 = AP^2 - AM^2$, or $AM^2 + BM^2 = AP^2 + BP^2$.

Conversely the last equation being true for any two points M and P, it is readily seen that the perpendicular L to AM, through M, will be cut harmonically by the circles whose radii are AP and BP. But if the last equation be true for any one position of P, it is true for all, since

$$AP^2 + BP^2 = 2\,(CP^2 + AC^2) = \textit{a constant}.$$

Hence if L cut any one pair of conjugate circles (A) and (B) harmonically, it will cut every other pair in like manner; in other words, the envelope of L is independent of P.

The envelope in question is a conic whose foci are A *and* B *and whose axis is the diameter of* (C).

For the last equation but one may also be written thus,

$$2\,(CM^2 + AC^2) = 2\,(CP^2 + AC^2),$$

so that CM = CP, and M is on the circle (C). The envelope of L, therefore, is the *first negative pedal* of the circle (C) relative to the origin A, and this is well known to be an ellipse when A is within (C), a hyperbola when A is without (C), and a pair of points when A is on (C).

[Mr. TOWNSEND remarks, that the theorem follows from the fact that the envelope in question is (see *Modern Geometry*, Chap. X.) the ellipse (or hyperbola) of which A and B are the foci and the circle on whose diameter they lie the director circle.]

II. *Solution by* PROFESSOR CAYLEY.

1. More generally, the problem may be thus stated:—If two conics touch at I, J the lines OI, OJ respectively; if P be a variable point on the first conic, and OAB a fixed line through O meeting the second conic in the points A and B; then considering the conic which passes through P and touches at I, J the lines AI, AJ respectively, and also the conic which passes through P and touches at I, J the lines BI, BJ respectively; the envelope of the lines which cut harmonically the last-mentioned two conics is a conic independent of the position of P.

2. Taking $x = 0$, $y = 0$, $z = 0$ for the equations of the lines OI, JI, and OJ respectively, the equations of the two given conics are

$$xz - y^2 = 0,\ kxz - y^2 = 0;$$

hence the coordinates of P may be taken to be

$$x : y : z = 1 : \theta : \theta^2.$$

and the coordinates of the points A and B may be taken to be

$$x : y : z = 1 : ka : ka^2, \text{ and } x : y : z = 1 : -ka : ka^2.$$

The equations of the lines AI, AJ are

$$kax - y = 0,\ z - ay = 0;$$

hence the equation of the conic touching these lines at the points I, J respectively, and also passing through the point P, is

$$\frac{(kax-y)\,(z-ay)}{(ka-\theta)\,(\theta-a)}=\frac{y^2}{\theta},$$

and similarly the equations of the lines BI, BJ being

$$kax+y=0,\ \ z+ay=0,$$

the equation of the conic touching these lines at the points I, J respectively, and also passing through the point P, is

$$\frac{(kax+y)\,(z+ay)}{(ka+\theta)\,(\theta+a)}=\frac{y^2}{\theta},$$

or multiplying out and reducing, if the equations of the two conics are represented by

$$(a,\ b,\ c,\ f,\ g,\ h)\,(x,\ y,\ z)^2=0,\ \ (a',\ b',\ c',\ f',\ g',\ h')\,(x,\ y,\ z)^2=0,$$

respectively, then the values of the coefficients are

$$\begin{array}{ll}
a=0, & a'=0\\
b=2\,(ka\ +\theta^2-ka\theta), & b'=2\,(-ka^2-\theta^2-ka\theta)\\
c=0, & c'=0\\
f=-\theta, & f'=\theta\\
g=\theta ka, & g'=\theta ka\\
h=-\theta ka^2, & h'=\theta ka^2.
\end{array}$$

Now the tangential equation of the envelope of the line which cuts harmonically the last-mentioned two conics, is

$$(bc'+b'c-2ff',\ .\ ,\ .\ gh'+g'h-af'-a'f,\ .\ ,\ .)\,(\xi,\ \eta,\ \zeta)^2\,;$$

or substituting for a &c. a' &c., their values, it is found that the coefficients of this equation have all of them the common factor $2\theta^2$, and that omitting this factor the equation is independent of θ, viz., the tangential equation of the envelope in question is

$$\left(1,\ -k^2a^2,\ k^2a^4,\ 0,\ k\,(2k-1)\,a^2,\ 0\right)\,(\xi,\ \eta,\ \zeta)^2=0,$$

which proves the theorem.

3. *In particular*, if $k=1$, that is if the points A, B lie on the conic $xz-y^2=0$, then the tangential equation of the envelope is

$$(1,\ -a^2,\ a^4,\ 0,\ a^2,\ 0)\,(\xi,\ \eta,\ \zeta)^2=0,$$

that is

$$\xi^2-a^2\eta^2+a^4\zeta^2+2a^2\xi\zeta=0\,;$$

or, what is the same thing, the equation is

$$(\xi-a\eta+a^2\zeta)\,(\xi+a\eta+a^2\zeta)=0,$$

and thus the envelope breaks up into the two points

$$\xi-a\eta+a^2\zeta=0,\ \xi+a\eta+a^2\zeta=0\,;$$

that is, the points $(1,\ -a,\ a^2)$ and $(1,\ a,\ a^2)$, which are the points A and B respectively. That is, in the problem in its original form, if the points A and B are the extremities of a diameter of a given circle, then the two constructed circles are a pair of orthotomic circles with the centres A and B respectively; and the theorem is the very obvious one, that any line through the centre of either circle cuts the two circles harmonically.

2220. (Proposed by W. K. Clifford, B.A.)—A, B, C, D are four points on a circle, and through every pair, as AB, another circle (AB) is drawn; then the pair of circles (AB), (CD) intersects the pair (AC), (DB) in four new points on a circle U; the pair (AC), (DB) meets (AD), (BC) on a circle V; and the pair (AD), (BC) meets (AB), (CD) on a circle W; also the three circles U, V, W have a common radical axis. (This may be extended to spheres; and these are also analogous properties of rectangular hyperbolas.)

Solution by the Rev. R. Townsend, F.R.S.

If X and X′, Y and Y′, Z and Z′, each $= 0$, be the equations of the three pairs of opposite circles (BC) and (AD), (CA) and (BD), (AB) and (CD) which intersect three by three at the same four points A, B, C, D on the same circle $S = 0$; then, constant multipliers properly selected being implicitly supposed, the three quartics $YY'-ZZ' = 0$, $ZZ'-XX' = 0$, $XX-YY' = 0$ will break up into the three pairs of circles $SU = 0$, $SV = 0$, $SW = 0$, from which, since identically $U+V+W = 0$, therefore, &c.

2254. (Proposed by Messrs. Hermite and Sylvester.)—If f be any quantic in (x, y), and $\phi = \left(\alpha x\dfrac{d}{dx} + ay\dfrac{d}{dx} + \beta x\dfrac{d}{dy} + by\dfrac{d}{dy}\right)f$, show that if $(\alpha-b)^2+4a\beta =$ or < 0 the number of real roots in ϕ is at least as great as in f.

Solution by Thomas Savage, M.A.

Let $f = x^n\,\psi\left(\dfrac{y}{x}\right) = x^n\,\psi(\nu)$ suppose; then it is easily seen that

$$\phi = x^n\left\{n(\alpha+a\nu)\,\psi + (\beta+b\nu-\alpha\nu-a\nu^2)\,\psi'\right\}.$$

Now, if $(\alpha-b)^2+4a\beta =$ or < 0, the factor which multiplies ψ' retains a constant sign, and therefore if for ν we substitute the several real roots of $f = 0$, or $\psi(\nu) = 0$, the signs of ϕ will always be the same as, or always the opposite to, the signs of $\psi'(\nu)$; i. e., the number of changes of sign of ϕ will be one less than the number of real roots of f or ψ. But these two equations are evidently of the same degree. Hence the number of real roots of ϕ and f can differ, if they differ at all, only by an even number. Hence the number of real roots of ϕ must be at least equal to those of f.

2258. (Proposed by M. W. Crofton, B.A.)—Prove that

$$\frac{\pi^2}{8} = \frac{3^2}{2.4}\cdot\frac{5^2}{4.6}\cdot\frac{7^2}{6.8}\cdot\frac{11^2}{10.12}\cdot\frac{13^2}{12.14}\cdot\frac{17^2}{16.18}\cdots\cdot$$

the numerators being the squares of the prime numbers.

Solution by the PROPOSER.

It is known that $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \&c.,$

$$\therefore\ \frac{\pi^2}{8} = \frac{1}{3^2}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \&c.\right) + 1 + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \&c.$$

therefore $\frac{\pi^2}{8}\ \frac{3^2-1}{3^2} = 1 + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \&c.,$

the denominators comprising no multiples of 3.

Again, $\frac{\pi^2}{8} \cdot \frac{3^2-1}{3^2} = \frac{1}{5^2}\left(1 + \frac{1}{5^2} + \frac{1}{7^2} + \&c.\right) + 1 + \frac{1}{7^2} + \frac{1}{11^2} + \&c.$

$$\therefore\ \frac{\pi^2}{8} \cdot \frac{3^2-1}{3^2} \cdot \frac{5^2-1}{5^2} = 1 + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{17^2} + \&c.,$$

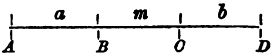
the denominators being prime to 3 and 5.

Again, $\frac{\pi^2}{8} \cdot \frac{3^2-1}{3^2} \cdot \frac{5^2-1}{5^2} \cdot \frac{7^2-1}{7^2} = 1 + \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{17^2} + \&c.,$

so that ultimately $\frac{\pi^2}{8} \cdot \frac{3^2-1}{3^2} \cdot \frac{5^2-1}{5^2} \cdot \frac{7^2-1}{7^2} \ldots\ldots = 1$, which proves the theorem.

2271. (Proposed by Professor SYLVESTER.)—A, B, C, D are four successive points in a straight line. If (I) two points be taken at random in AD, and one point at random in BC, find the chance that the latter will lie between the two former. And again, if (II) two points be taken at random in BC, and one point at random in AD, find the chance that the latter will lie between the two former.

Solution by M. W. CROFTON, B.A.



I. Let AB $= a$, BC $= m$, CD $= b$, AD $= l$. Let x, y be the distances from A of two points taken at random in AD, z the distance from A of the third point, taken in BC. The measure (F) of the favourable cases will be the integral $\iiint dx\,dy\,dz$ extended through all values which give z intermediate in value between x and y. Also it will be sufficient to take the cases where $x > y$, and double the result.

Hence, as $x > z > y$, $l > x$, $y > 0$, it will be easy to see that

$$F = 2\int_a^{a+m} dz \int_z^l dx \int_0^z dy = \frac{m^3}{3} + (a+b)\,m^2 + 2abm\,;$$

and the whole number of cases being ml^2, the required probability is

$$p = \frac{1}{l^2}\left(\frac{m^2}{3} + (a+b)\,m + 2ab\right).$$

If $a = b = 0$, we have $p = \frac{1}{3}$, which is the chance of any particular one of three points, all taken at random in a given line, lying between the other two. This is evident à priori, as any one is as likely as another to be the middle one.

The above result may also be obtained thus. Let the random points be called 1, 2, 3. (1.) If 1 and 2 are in BC; the chance of this is $\frac{m^2}{l^2}$, and if it is so, the chance of 3 being the middle one is $\frac{1}{3}$; hence the whole probability arising from this case is $\frac{m^2}{3l^2}$. (2.) If 1 or 2 is in AB, and the other in CD; the chance of this is $\frac{2ab}{l^2}$; and this is the whole probability from this case; as 3 is necessarily the middle point. (3.) If 1 or 2 is in AB or CD, the other being in BC. Suppose 1 to fall on AB or CD, the chance of this is $\frac{a+b}{l}$, while the chance of 2 falling on BC is $\frac{m}{l}$; and if 1 and 2 fall in this way, it is clearly an even chance that 3 lies between them, so that the whole chance is $\frac{(a+b)\,m}{2l^2}$; now this must be doubled, as 2 may fall on AB or CD, and 1 on BC. Therefore the probability from this case is $\frac{(a+b)\,m}{l^2}$. The above cases being mutually exclusive, we have as before

$$p = \frac{m^2}{3l^2} + \frac{2ab}{l^2} + \frac{(a+b)\,m}{l^2}.$$

II. If 1 and 2 are taken in BC, and 3 in AD, the chance that 3 is in BC is $\frac{m}{l}$; and if so, the chance of its lying in the middle is $\frac{1}{3}$. Hence $p = \frac{m}{3l}$.

This may also be found from the expression

$$p = \frac{1}{m^2 l} . 2 \int_a^{a+m} dz \int_z^{a+m} dx \int_a^z dy.$$

2277. (Proposed by F. D. Thomson, M.A.)—Through a point O straight lines AO, BO, CO are drawn to meet the sides of the triangle ABC in D, E, F. Find the locus of O in order that the three perpendiculars from A, B, C on EF, FD, DE respectively may meet in a point.

I. *Solution by* S. Roberts, M.A.; H. Tomlinson; W. Chadwick; E. McCormick; J. Dale; *and others.*

Take the triangle ABC as the triangle of reference, and let (α, β, γ) be the the coordinates of O. Then the equations of EF, FD, DE are of the forms

$$\frac{x}{\alpha} - \frac{y}{\beta} - \frac{z}{\gamma} = 0, \quad \frac{y}{\beta} - \frac{x}{\alpha} - \frac{z}{\gamma} = 0, \quad \frac{z}{\gamma} - \frac{x}{\alpha} - \frac{y}{\beta} = 0.$$

We have to take the lines $my-nz=0$, $nx-lz=0$, $ly-mx=0$ respectively perpendicular to EF, FD, DE. But this requires that

$$n\left(\frac{1}{\beta}-\frac{\cos A}{\gamma}+\frac{\cos C}{\alpha}\right)=m\left(\frac{1}{\gamma}-\frac{\cos A}{\beta}+\frac{\cos B}{\alpha}\right),$$

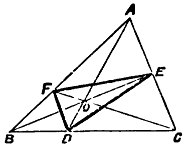
$$m\left(\frac{1}{\alpha}-\frac{\cos C}{\beta}+\frac{\cos B}{\gamma}\right)=l\left(\frac{1}{\beta}-\frac{\cos C}{\alpha}+\frac{\cos A}{\gamma}\right),$$

$$l\left(\frac{1}{\gamma}-\frac{\cos B}{\alpha}+\frac{\cos A}{\beta}\right)=n\left(\frac{1}{\alpha}-\frac{\cos B}{\gamma}+\frac{\cos C}{\beta}\right).$$

When l, m, n are eliminated and the factors are multiplied out, we have for the locus required the cubic

$$\begin{aligned}&\alpha(\beta^2-\gamma^2)\cos A+\beta(\gamma^2-\alpha^2)\cos B+\gamma(\alpha^2-\beta^2)\cos C\\&\qquad+(\gamma\cos C-\beta\cos B)(\beta\cos B-\alpha\cos A)(\alpha\cos A-\gamma\cos B)=0.\end{aligned}$$

II. *Solution by the* PROPOSER.



Using trilinear coordinates, and taking ABC as triangle of reference, if O is given by $lx=my=nz$, the equation to EF is $-lx+my+nz=0$.

Let the equation to the perpendicular from A on EF be $\mu y-\nu z=0$; then

$$m\mu-n\nu-(-m\nu+\mu n)\cos A-l\nu\cos B+l\mu\cos C=0,$$

or $$\mu\left\{m-n\cos A+l\cos C\right\}=\nu\left\{n-m\cos A+l\cos B\right\};$$

hence the equations to the three perpendiculars are

$$\left\{n-m\cos A+l\cos B\right\}y=\left\{m-n\cos A+l\cos C\right\}z,$$

$$\left\{l-n\cos B+m\cos C\right\}z=\left\{n-l\cos B+m\cos A\right\}x,$$

$$\left\{m-l\cos C+n\cos A\right\}x=\left\{l-m\cos C+n\cos B\right\}y;$$

and these will meet in a point if

$$\begin{aligned}&\left\{n-m\cos A+l\cos B\right\}\left\{l-n\cos B+m\cos C\right\}\left\{m-l\cos C+n\cos A\right\}\\&=\left\{m-n\cos A+l\cos C\right\}\left\{n-l\cos B+m\cos A\right\}\left\{l-m\cos C+n\cos B\right\}.\end{aligned}$$

After a good deal of reduction this equation becomes

$$mn\sin^2 A\left\{m\cos C-n\cos B\right\}+\&c.=0.$$

But if (x, y, z) be coordinates of O, we have $x:y:z=l^{-1}:m^{-1}:n^{-1}$, hence the equation of the locus of O becomes

$$\frac{\sin^2 A}{yz}\left\{\frac{\cos C}{y}-\frac{\cos B}{z}\right\}+\&c.=0,$$

or $$\begin{aligned}&x^2\sin^2 A\left\{z\cos C-y\cos B\right\}+y^2\sin^2 B\left\{x\cos A-z\cos C\right\}\\&\qquad+z^2\sin^2 C\left\{y\cos B-x\cos A\right\}=0\;\ldots\ldots\text{(i.)}\end{aligned}$$

or $$yz\left\{z\cos B\sin^2 C-y\cos C\sin^2 B\right\}+\&c.=0\;\ldots\ldots\ldots\text{(ii.)},$$

therefore the locus of a *cubic* through A, B, C, G the centre of gravity, T the intersection of perpendiculars to the angular points of the triangle PQR formed by drawing lines through ABC parallel to the opposite sides, and the point S given by

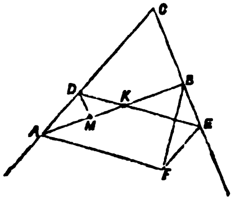
$$\frac{x \sin^2 A}{\cos A} = \frac{y \sin^2 B}{\cos B} = \frac{z \sin^2 C}{\cos C} \ldots\ldots (S).$$

Also it may be shown that T is the *satellite* of A, B, C, S,
and S the satellite of P, Q, R, G.

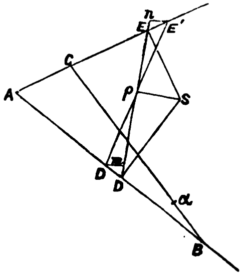
2282. (Proposed by the Rev. R. Townsend, F.R.S.)—Draw the minimum chord of a given angle which can be cut in a given ratio by a given line.

I. *Solution by* H. Murphy; E. McCormick; *and others.*

Let AB be the given line, and DKE the minimum line required; complete the parallelogram ADEF, draw DM parallel to BE, and join BF. Then BE : DM = KE : KD is given, and DM : DA = BC : CA is also given; therefore BE : DA, or BE : EF is given; moreover the angle FEC being the supplement of C, is given. Hence the species of the triangle FBE and the position of the line BF are given; and the minimum line from A on BF is the perpendicular, whence the construction is obvious.



II. *Solution by* Morgan Jenkins, B.A.



Let BC be the segment of the given line cut off by the given angle A; then, representing the sides of the triangle ABC by u, v, w respectively, and one of the variable lines by θ, we have the anharmonic ratio ($\theta \;\; wuv \infty$) is equal to a given ratio; therefore θ envelopes a conic touching u, v, w, and ∞, that is, a parabola touching the sides of the triangle ABC. Again, let DE, D′E′ be two consecutive chords of the angle A; let them meet in ρ; and draw E′n, D′m perpendicular to DE. Then if DE be a minimum, we have in the limit DE = D′E′, $\rho n = \rho$E′, and $\rho m = \rho$D′; therefore Dm = En, or EE′ cos E = DD′ cos D; hence ρD cot D = ρE cot E. Since DE is a tangent to the parabola, it is evident that it is the tangent at the vertex;

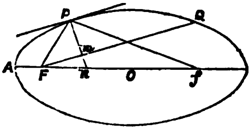
for if ρ be the vertex, S the focus, $S\rho$ is perpendicular to DE, and SE, SD to AE, AD, therefore $S\rho = \rho D \cot ADE = \rho E \cot AED$.

Again, if α, β, γ be the points of contact of the parabola with the sides of the triangle, $\frac{B\alpha}{C\alpha} = \frac{AC}{C\beta} = \frac{B\gamma}{AB}$ = a given ratio. The focus S can be found by known theorems from the three tangents and any one of the points of contact. Then drawing SD, SE perpendicular to AB, AC respectively, DE is the chord required.

2298. (Proposed by R. Tucker, M.A.)—If r_1, r_2 be a pair of focal *covectors* of an ellipse, and ϕ_1 the angle between them, then prove that
$r_2(1-\cos\phi_1) = p$, where p is the semi-parameter.

Def.—By *covectors* I mean a system of vectors so taken that the nth is parallel to the tangent at the extremity of the $(n-1)$th.

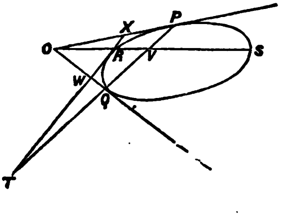
Solution by S. Watson; Rev. J. L. Kitchin, M.A.; H. Tomlinson; J. Dale; E. McCormick; *and others.*



Let F, f be the foci; FP $(=r_1)$ and FQ $(=r_2)$ the covectors; bisect the angle FPf by the line Pn, cutting FQ perpendicularly in m, and put $\angle$ QF$f = \alpha$. Then
$r_2(1-e\cos\alpha) = p$ (semi-parameter).... (1).
Also, $FP : Pf = Fn : nf$, $\therefore$ $FP : 2a = Fn : 2ae$; hence
$$e\,.\,FP = Fn = Fm\,.\sec\alpha = FP\,.\cos\phi_1\sec\alpha,\ \therefore\ e\cos\alpha = \cos\phi_1,$$
and (1) becomes $r_2(1-\cos\phi_1) = p$, the relation required.

2248. (Proposed by W. Chadwick.)—Prove that any tangent to a conic is divided harmonically by the point of contact and the three points where it meets any two tangents and their chord of contact.

I. *Solution by* F. D. Thomson, M.A.; W. H. Laverty; E. McCormick; J. Dale; *the* Proposer; *and others.*



Let OP, OQ, PQ be the two tangents and their chord of contant, TWRX the third tangent; then we have
(O . TWRX) = (R . TQVP)
= (R . RQSP) = (P . RVSO);
but (RVSO) is harmonic, $\therefore$ (TWRX) is harmonic. Where TX is an asymptote, R moves off to infinity and WX is bisected in T; hence the segment intercepted on an asymptote to a conic by two tangents is bisected by the chord of contact.

II. *Solution by* Archer Stanley.

The theorem is a particular case of **Desargues'** Theorem, according to which every transversal is cut in three pairs of points in involution by a conic and any two pairs of opposite sides of a quadrangle ABCD inscribed therein. In fact, if we suppose A and B to coincide, as well as C and D, then the opposite sides AB and CD of the quadrangle will become tangents and the other pair of opposite sides AC and BD will coincide with the chord of contact of these tangents ; consequently, if the chord cut the transversal in *e*, and we moreover assume that this transversal, instead of cutting, *touches* the conic in *f*, *e* and *f* will be the *double points* of the involution determined thereon, and as such they will, by a well known property, divide every segment of the involution harmonically.

A particular case of our theorem is the well known one *that the segment intercepted on any tangent to a hyperbola by the asymptotes is bisected by its points of contact.* The latter being assumed, therefore, the former follows by simple projection.

2010. (Proposed by the Editor.)—If Q, R be the foci of an ellipse inscribed in a triangle ABC, prove that

$$\mathrm{AQ}.\mathrm{AR}.\mathrm{BC}+\mathrm{BQ}.\mathrm{BR}.\mathrm{CA}+\mathrm{CQ}.\mathrm{CR}.\mathrm{AB}=\mathrm{BC}.\mathrm{CA}.\mathrm{AB}.$$

Solution by Omega.

Let the triangle ABC be taken as triangle of reference, and let (α, β, γ) be the coordinates of the focus Q; then (see *Messenger of Mathematics*, Vol I., pp. 99), the coordinates of R will be

$$\frac{k^2}{\alpha},\ \frac{k^2}{\beta},\ \frac{k^2}{\gamma},\ \text{where } \frac{k^2}{\alpha\beta\gamma}=\frac{2\Delta}{a\beta\gamma+b\gamma\alpha+c\alpha\beta};$$

$$\therefore\ \mathrm{AQ}^2=\frac{\beta^2+\gamma^2+2\beta\gamma\cos\mathrm{A}}{\sin^2\mathrm{A}},\ \text{and } \mathrm{AR}^2=\frac{k^4}{\beta^2\gamma^2}\cdot\frac{\beta^2+\gamma^2+2\beta\gamma\cos\mathrm{A}}{\sin^2\mathrm{A}};$$

$$\therefore\ \mathrm{AQ}.\mathrm{AR}.\sin\mathrm{A}=\frac{k^2}{\beta\gamma}\cdot\frac{\beta^2+\gamma^2+2\beta\gamma\cos\mathrm{A}}{\sin\mathrm{A}}=\frac{2\Delta\alpha}{\sin\mathrm{A}}\cdot\frac{\beta^2+\gamma^2+2\beta\gamma\cos\mathrm{A}}{a\beta\gamma+b\gamma\alpha+c\alpha\beta}$$

$$=\frac{bc\alpha(\beta^2+\gamma^2+2\beta\gamma\cos\mathrm{A})}{a\beta\gamma+b\gamma\alpha+c\alpha\beta}=\frac{bc\alpha\beta^2+bc\alpha\gamma^2+(b^2+c^2-a^2)\alpha\beta\gamma}{a\beta\gamma+b\gamma\alpha+c\alpha\beta},$$

and we have similar expressions for BQ . BR . sin B and CQ . CR . sin C.

Therefore, by addition, we have

$$\mathrm{AQ}.\mathrm{AR}.\sin\mathrm{A}+\mathrm{BQ}.\mathrm{BR}.\sin\mathrm{B}+\mathrm{CQ}.\mathrm{CR}.\sin\mathrm{C}$$

$$=\frac{bc\alpha(\beta^2+\gamma^2)+ca\beta(\gamma^2+\alpha^2)+ab\gamma(\alpha^2+\beta^2)+(a^2+b^2+c^2)\alpha\beta\gamma}{a\beta\gamma+b\gamma\alpha+c\alpha\beta}$$

$$=a\alpha+b\beta+c\gamma,\ \text{(by division)}=abc\,\frac{\sin\mathrm{A}}{\mathrm{BC}}=abc\,\frac{\sin\mathrm{B}}{\mathrm{CA}}=abc\,\frac{\sin\mathrm{C}}{\mathrm{AB}},$$

$$\therefore\ \mathrm{AQ}.\mathrm{AR}.\mathrm{BC}+\mathrm{BQ}.\mathrm{BR}.\mathrm{CA}+\mathrm{CQ}.\mathrm{CR}.\mathrm{AB}=\mathrm{BC}.\mathrm{CA}.\mathrm{AB}.$$

2315. (Proposed by the Rev. J. Blissard.)—Sum the series

$$(1)\ldots\ldots \frac{x}{1}-\frac{x^4}{4}+\frac{x^7}{7}-\&c., \quad (2)\ldots\ldots \frac{x}{1}-\frac{x^5}{5}+\frac{x^9}{9}-\&c.$$

Solution by S. Watson; H. Tomlinson; E. McCormick; *and many others.*

Generally the series

$$\frac{x}{1}\pm\frac{x^{m+1}}{m+1}+\frac{x^{2m+1}}{2m+1}\pm\&c.=\int_0^x\frac{dx}{1\mp x^m}\ldots\ldots(A).$$

(1.) In this case $m=3$, and (A) becomes (using the lower sign)

$$\int_0^x\frac{dx}{1+x^3}=\tfrac{1}{3}\int_0^x\left(\frac{1}{1+x}-\frac{x-2}{1-x+x^2}\right)dx$$

$$=\tfrac{1}{6}\log\left(\frac{(1+x)^2}{1-x+x^2}\right)+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x\sqrt{3}}{2-x}\right).$$

(2). In this case $m=4$, and (A) becomes, as before,

$$\int_0^x\frac{dx}{1+x^4}=\frac{1}{2\sqrt{2}}\int_0^x\left(\frac{x+\sqrt{2}}{1+x\sqrt{2}+x^2}-\frac{x-\sqrt{2}}{1-x\sqrt{2}+x^2}\right)dx$$

$$=\frac{1}{4\sqrt{2}}\log\frac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2}+\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x\sqrt{2}}{1-x^2}\right).$$

2335. (Proposed by the Rev. J. Blissard.)—Sum the series

$$(1)\ldots\frac{1}{1.4}+\frac{1}{5.8}+\frac{1}{9.12}+\&c., \quad (2)\ldots\frac{1}{1.3}+\frac{1}{7.9}+\frac{1}{13.15}+\&c.$$

Solution by Stephen Watson; E. McCormick; H. Tomlinson; *and many others.*

Generally the series

$$\frac{1}{m\,(m+n)}+\frac{1}{(m+r)\,(m+n+r)}+\frac{1}{(m+2r)\,(m+n+2r)}+\&c.$$

$$=\frac{1}{n}\left\{\frac{1}{m}+\frac{1}{m+r}+\&c.-\left(\frac{1}{m+n}+\frac{1}{m+n+r}+\&c.\right)\right\}$$

$$=\frac{1}{n}\int_0^1\left\{\frac{x^{m-1}}{1-x^r}-\frac{x^{m+n-1}}{1-x^r}\right\}dx$$

$$=\frac{1}{n}\int_0^1\frac{x^{m-1}\,(1-x^n)}{1-x^r}\,dx\ldots\ldots\ldots\ldots\ldots\ldots(A).$$

(1.) In this case $m = 1$, $n = 3$, $r = 4$, and (A) becomes

$$\tfrac{1}{3}\int_0^1 \frac{1+x+x^2}{(1+x)(1+x^2)}\,dx = \tfrac{1}{4}\log 2 + \frac{\pi}{24}.$$

(2.) In this case $m = 1$, $n = 2$, $r = 6$, and (A) becomes

$$\tfrac{1}{2}\int_0^1 \frac{dx}{1+x^2+x^4} = \tfrac{1}{4}\int_0^1 \left\{\frac{1+x}{1+x+x^2} + \frac{1-x}{1-x+x^2}\right\} dx$$

$$= \tfrac{1}{8}\log 3 + \frac{1}{4\sqrt{3}}\left(\frac{\pi}{3}+\frac{\pi}{6}\right) = \tfrac{1}{8}\log 3 + \frac{\pi\sqrt{3}}{24}.$$

2135. (Proposed by W. K. CLIFFORD, B.A.)—A tangent to an ellipse is a chord of a concentric circle, whose radius is equal to the distance between the ends of the axes of the ellipse; show that the straight lines which join the ends of the chord to the centre are conjugate diameters.

Solution by S. WATSON; R. TUCKER, M.A.; *the* PROPOSER; *and others.*

Let the equations to the ellipse, the circle, and the tangent to the ellipse at the point (h, k) on the curve be, respectively,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\;\; x^2+y^2=a^2+b^2,\;\; \frac{hx}{a^2}+\frac{ky}{b^2}=1 \ldots\ldots (1, 2, 3);$$

then the intersections of the tangent with the circle will be given by the simultaneous equations (2, 3); hence, eliminating y and x separately from these equations, and putting (x_1, x_2), (y_1, y_2) for the roots of the resulting equations, we have

$$\frac{x_1x_2}{a^4} = \frac{b^4-(a^2+b^2)k^2}{a^4k^2+b^4h^2},\text{ and } \frac{y_1y_2}{b^4} = \frac{a^4-(a^2+b^2)h^2}{a^4k^2+b^4h^2};$$

therefore $\quad a^2y_1y_2+b^2x_1x_2 = 0,\;$ or $\;\dfrac{y_1y_2}{x_1x_2} = -\dfrac{b^2}{a^2};$

hence the lines joining the centre with the points (x_1, y_1), (x_2, y_2), that is with the ends of the chord, are conjugate diameters.

[If we change b^2 into $-b^2$ we obtain a similar theorem for the hyperbola, the square on the radius of the circle being equal to the *sum* of the squares on the semi-axis in the ellipse and to their *difference* in the hyperbola; but the conjugate diameters will be imaginary if

$$(a^4k^2+b^4h^2)(a^2-b^2) < a^4b^4].$$

2262. (Proposed by ALIQUIS.)—Evaluate the definite integral

$$\int_0^\pi \log \sin \theta \,.\, d\theta.$$

Solution by the PROPOSER ; H. TOMLINSON ; J. DALE ; *and others.*

Putting D for the value of the integral, we have

$$D = \tfrac{1}{2}\int_0^\pi d\theta \log(1-\cos^2\theta) = -\tfrac{1}{2}\int_0^\pi d\theta\,(\cos^2\theta + \tfrac{1}{2}\cos^4\theta + \tfrac{1}{3}\cos^6\theta + \&c.),$$

or
$$-\frac{2}{\pi}D = \frac{1}{2} + \frac{1}{2}\cdot\frac{1.3}{2.4} + \frac{1}{3}\cdot\frac{1.3.5}{2.4.6} + \&c.$$

$$= \int_0^1 dx\,\frac{1}{x}\left(\frac{1}{\sqrt{(1-x)}}-1\right) = {}_0^1\left[\log\frac{x}{1-\frac{1}{2}x+\sqrt{(1-x)}} - \log x\right]$$

$$= {}_1^0\left[\log\left\{1-\tfrac{1}{2}x+\sqrt{(1-x)}\right\}\right] = \log 2 - \log\tfrac{1}{2} = -2\log\tfrac{1}{2};$$

therefore
$$D \equiv \int_0^\pi \log\sin\theta\,.\,d\theta = \pi\log\tfrac{1}{2}.$$

[Another method of evaluating this definite integral is given in Todhunter's *Integral Calculus*, Art. 51.]

1982. (Proposed by the Rev. R. H. WRIGHT, M.A.)—A cylindrical vessel containing fluid revolves round a vertical axis: required the form which the fluid will assume. If the vessel be full and be whirled round the same axis, find the whole pressure against the side, also find the whole pressure on the top of the cylinder.

Solution by J. DALE ; E. MCCORMICK ; *the* PROPOSER ; *and others.*

Taking the axis of the vessel for the axis of z (z being measured downwards) and its top for the plane of xy, and putting ρ for the density of the fluid, the general equation for determining the pressure p at any point is $dp = \rho\,(Xdx + Ydy + Zdz)$, where X, Y, Z are the forces parallel to the axes; and if ω = angular velocity, r = distance of any point from the axis of revolution, and g = force of gravity, we have $X = \omega^2 x$, $Y = \omega^2 y$, $Z = g$,

$$\therefore\ dp = \rho\left\{\omega^2(xdx+ydy)+gdz\right\},\ \text{whence}\ p = \rho\,(\tfrac{1}{2}\omega^2r^2+gz) + \text{constant},$$

showing that the surfaces of equal pressure are paraboloids of revolution about the axis of z.

To find the pressure on the side when the vessel is full; let a = radius, and

h = depth ; then annular element of surface $= 2\pi a dz$,
and pressure on element $= \rho\,(\frac{1}{2}\omega^2 a^2 + gz)$,

$$\therefore \text{ whole pressure } = \int_0^a 2\pi a\rho\,(\tfrac{1}{2}\omega^2 a^2 + gz)\,dz = \pi\rho a^3\omega^2 h + \pi\rho a g h^2,$$

the first term expressing the pressure on the side due to the rotation, and the last the pressure due to gravity.

To find the pressure on the top, let as before r = distance of any point from the axis, then element of top $= 2\pi r dr$, and pressure on element $= \frac{1}{2}\rho\omega^2 r^2$,

therefore the whole pressure $= \int_0^a \pi\rho\omega^2 r^3 dr = \frac{1}{4}\pi\rho\omega^2 a^4$.

1962. (Proposed by W. K. CLIFFORD, B.A.)—Required the *characteristics* of the system of conics having five-pointic contact with a curve of order m and class n.

Solution by PROFESSOR CREMONA.

M. ZEUTHEN a démontré dans son mémoire *Nyt Bidrag til Læren om Systemer af Keglesnit* (Kjöbenhavn, 1865), p. 88, que les caractéristiques du système des coniques qui ont un contact cinq-ponctuel avec une courbe d'ordre m et de classe n, ayant κ rébroussements et ι inflexions, sont

$$\mu = \tfrac{2}{3}(5\iota + 4\kappa) = 10(n-m) + 6\kappa = 8(m-n) + 6\iota,$$
$$\nu = \tfrac{2}{3}(5\kappa + 4\iota) = 10(m-n) + 6\iota = 8(n-m) + 6\kappa.$$

2115. Proposed by T. COTTERILL, M.A.)—1. The line joining any two of the centres A, B, C, D of four circles (A), (B), (C), (D) cutting each other at right angles in a plane, is perpendicular to the line joining the other two; any circle (A) cuts the sides of the triangle BCD harmonically, and the square of its radius varies inversely as the area of the triangle BCD. The sum of the squares of the radii is equal to the square of the diameter of the equal circles circumscribing the triangles, and the sum of the inverse squares is nothing.

2. If the circle described on AB as a diameter be denoted by (AB), then the coaxal system of circles (A) (B) (AB) and the line CD are orthogonal to the coaxal system of circles (C) (D) (CD) and the line AB.

3. The locus of the real foci lying on lines parallel to AB of conics inscribed in the triangle formed from the intersection of the lines (BC, AD), (CA, BD) and (AB, CD) is the circle (CD) and the line AB.

Solution by the PROPOSER; J. DALE; *and others.*

1. To find the relative position of the centres A, B, C, D and the magnitudes of the radii ρ_a, ρ_b, ρ_c, ρ_d of four circles mutually orthogonal, we have the following system of equations:—

$$AB^2 = \rho_a^2+\rho_b^2 : BC^2 = \rho_b^2+\rho_c^2 : CA^2 = \rho_c^2+\rho_a^2$$

$$CD^2 = \rho_c^2+\rho_d^2 : AD^2 = \rho_a^2+\rho_a^2 : BD^2 = \rho_b^2+\rho_d^2.$$

Hence $AB^2+CD^2=BC^2+AD^2=CA^2+BD^2$. But $2AB.CD \cos(AB, CD) = AD^2+BC^2-AC^2-BD^2$; which vanishes as well as cos (BC, AD) and cos (CA, BD) for the points in question. Hence each point is the intersection of the perpendiculars in the triangle formed from the other three, or is the centre of the circle touching the sides of the triangle XYZ formed by the intersection of the pairs of lines (BC, AD) (CA, BD) and (AB, CD); and we can therefore make use of the known trigonometrical properties of the figure. The same equations also give us, $\rho_a^2 = \frac{1}{2}(BD^2+CD^2-AD^2) = DB.DC \cos(BDC) = DC.DA \cos(CDA) = DA.DB \cos(ADB)$. But the equation to the circle, centre D and radius ρ_d is $DP^2-\rho_d^2 = 0$, and if P and Q are harmonic conjugates to it, $DP.DQ \cos(PDQ)-\rho_d^2 = 0$, which is satisfied by the pairs of points (B, C) (C, A) and (A, B). Hence the circle, centre D and radius ρ_d cuts each of the sides of the triangle ABC harmonically, and similarly for the rest. Now, if unity be taken as the radius of the circle circumscribing each of the triangles, and A, B, C are the angles of the triangle ABC, we have

$$\rho_a^2 = 4 \sin B \sin C \cos A;\ \rho_b^2 = 4 \sin C \sin A \cos B;$$

$$\rho_c^2 = 4 \sin A \sin B \cos C;\ \rho_d^2 = -4 \cos A \cos B \cos C;$$

$$DBC = 2 \cos B \cos C \sin A;\ DCA = 2 \cos C \cos A \sin B;$$
$$DAB = 2 \cos A \cos B \sin C;\ ABC = 2 \sin A \sin B \sin C;$$

and by multiplying the corresponding quantities, we see that the square of each radius varies inversely as the corresponding triangle, and that

$\Sigma\rho_a^2 = 4$, and $\Sigma\rho_a^{-2} = 0$. These expressions and other different relations can also be obtained by observing that $\rho_a^2 = 2r_a$; $\rho_b^2 = 2r_b$; $\rho_c^2 = 2r_c$; $\rho_d^2 = -2r_d$, where r_a, r_b, r_c, r_d are the radii of the circles touching the sides of the triangle XYZ, with centres A, B, C, D.

2. The circle (AB) described on the diameter AB is coaxal with the orthogonal circles (A), (B), because the tangents to each of the latter at a point of intersection is a radius of the other; and the line CD is coaxal, since the circles (C), (D) are orthogonal to the circles (A), (B); and similarly the system (C), (D), (CD) and line AB is coaxal, and the two systems are orthogonal, since two of the former are orthogonal to two of the latter. (TOWNSEND'S *Modern Geometry*, Art. 185, &c.)

3. Two points P, Q are foci of a conic inscribed in the triangle XYZ, if XC, XD, the bisectors of the angle X, bisect the angle PXQ, and YC, YD, the bisectors of the angle Y, bisect the angle PYQ. But, if P and Q are points on the circle (CD) such that PQ is parallel to AB, and therefore perpendicular to CD, or are points on AB harmonic conjugates to AB, this pro-

perty holds good. Hence the circle (CD) and the line AB is the locus of the real foci of conics inscribed in the triangle XYZ, whose axes are parallel to AB.

It is worth noticing, that P, Q are the points referred to by Professor SYLVESTER in Quest. 1990 (1), and in the present figure are the foci of conics touching either of the circles (A), (B) at their intersections.

2232. (Proposed by M. DUPAIN.)—Prove that the axes of the parabolas which have a common focus, and pass through two given points, are parallel to the asymptotes of the hyperbola which passes through the common focus, and has the two given points for foci.

I. *Solution by* PROFESSOR SYLVESTER.

Let F, G be the foci of the hyperbola whose semi-axis is a and eccentricity e, P any point in the curve. Let LM be the directrix of either parabola having P for focus and containing F and G. Draw FL, GM perpendiculars to LM. Then MG is the direction of the axis of the parabola. But

$$\sec \text{MGF} = \frac{\text{GF}}{\text{GM}-\text{FL}} = \frac{\text{GF}}{\text{GP}-\text{FP}} = \frac{2ae}{2a} = e, \text{ or } \tan \text{MGF} = \sqrt{(e^2-1)} = \pm\frac{b}{a};$$

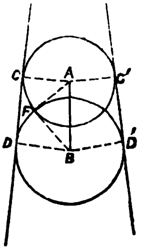
therefore the axis of the parabola is parallel to either asymptote. The distance of the directrix of the parabola from the centre of the hyperbola is obviously the mean of the distances of the foci of the hyperbola from the focus of the parabola.

It may be noticed that M. DUPAIN's theorem is only a particular case of the familiar one that if F, G be the foci, and P, Q any two points of a conic, we may construct a pair of hyperbolas passing through F, G, and having P, Q for foci; for if $\text{FP} \pm \text{GP} = \text{FQ} \pm \text{GQ}$, then $\text{PF}-\text{QF} = \mp(\text{PG}-\text{QG})$. Taking Q the infinite point on either asymptote of the conic, either hyperbola becomes a parabola having P for focus and its axis parallel to the asymptote containing the point so taken.

Geometrically considered, the problem of constructing the parabolas in the above question is that of drawing common tangents to two circles having F, G as centres and ρ, ρ' (i.e. FP, GP) for radii. There are therefore two other imaginary parabolas whose directrices are at the distance $\frac{1}{2}(\rho-\rho')$, or a, from the centre, and which make angles $\pm \sec^{-1}\dfrac{2ae}{\rho+\rho'}$ with the axis of the hyperbola. Contrariwise, if P be taken a point not in a hyperbola but in an ellipse instead, the axes of the real parabolas make variable angles $\pm \sec^{-1}\dfrac{2ae}{\rho-\rho'}$ with the axis of the ellipse, but the directrices are at the constant distance $\frac{1}{2}(\rho+\rho')$, or a, from the centre; *i.e.*, the directrices of such parabolas touch the circle described upon the major axis of the ellipse as diameter. There are besides for each position of P two imaginary parabolas with axes parallel to the asymptotes of the ellipse and directrices at varying distances $\frac{1}{2}(\rho-\rho')$ from its centre. There is thus a simple relation between the two cases appertaining to the ellipse and hyperbola respectively, the real parabolas in the one case having their directrices always parallel to a pair of fixed lines, and in the other always tangential to a fixed circle, and *vice versâ* for the imaginary parabolas.

In the fact of the existence of such cross analogy lies the chief interest of M. DUPAIN's elegant question.

II. *Solution by* ARCHER STANLEY; J. M. WILSON, M.A., F.G.S.; F. D. THOMSON, M.A.; J. DALE; *and others.*

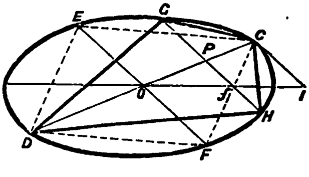


Let F be the common focus, A and B the two given points. Around A and B describe two circles passing through F. The common tangents CD and C′D′ of these circles are obviously the directrices of the two parabolas alluded to. Further, since the difference of the distances from A and B of each of these common tangents is equal to the difference of the distances of F from A and B, it is also equal to the transverse axis of the hyperbola in question; whence by a well known theorem we conclude that CD and C′D′ are perpendicular to its asymptotes. The latter, therefore, are parallel to the axes of the parabolas, since both are perpendicular to the same two lines.

[Mr. TOWNSEND gives the proof as follows:—"The property appears immediately from the two considerations that, if from any two points on a parabola perpendiculars be drawn to the axis, the distance between their feet is equal to the difference of the focal radii of the points, and that, if from the two foci of an hyperbola perpendiculars be drawn to either asymptote, the distance between their feet is equal to the difference of the focal radii of any point on the curve."]

2256. (Proposed by R. TUCKER, M.A.)—A quadrilateral in an ellipse has one diagonal a diameter of the curve, and the other a focal chord parallel to the conjugate diameter; show that its area varies as the conjugate diameter; also find the locus of the intersection of the diagonals.

Solution by S. WATSON; W. CHADWICK; H. TOMLINSON; W. H. LAVERTY; J. DALE; E. McCORMICK; *the* PROPOSER; *and others.*



Let COD, EOF be a diameter and its conjugate; GfH a focal chord parallel to EF, cutting OC in P; and CI a tangent at C meeting the major axis in I. Denote the point C by (x', y'); and, as usual, let $OC = a'$, $OE = b'$; then we have

$b = (a^2 - e^2x'^2)^{\frac{1}{2}}$, $\mathrm{OI} = \dfrac{a^2}{x'}$, and quad. CEDF $= 2ab$. Now

$$\frac{\text{quad. CGDH}}{\text{quad. CEDF}} = \frac{\mathrm{PG}}{\mathrm{OE}} = \left\{1 - \left(\frac{\mathrm{OP}}{\mathrm{OC}}\right)^2\right\}^{\frac{1}{2}} = \left\{1 - \left(\frac{\mathrm{O}f}{\mathrm{OI}}\right)^2\right\}^{\frac{1}{2}} = \frac{(a^2 - e^2x'^2)^{\frac{1}{2}}}{a} = \frac{b'}{a},$$

therefore quad. CGDH $= 2bb'$, which proves the first part of the question.

To find the locus of P we have to eliminate $\dfrac{x'}{y'}$ from the equations of OC and GH, that is to say, from the equations

$$y = \frac{y'}{x'}\,x, \text{ and } y = -\frac{b^2x'}{a^2y'}\,(x - ae);$$

the resulting equation is $a^2y^2 + b^2x\,(x - ae) = 0$, which is that of an ellipse whose major axis is $\mathrm{O}f$. [That the locus of P is an ellipse passing through O and F, wherever the fixed point f may be, is evident from the fact that P is the intersection of the corresponding rays of two homographic pencils whose vertices are O and f.]

2263. (Proposed by HUGH GODFRAY, M.A.)—Two chords are drawn at random in a circle; find the chance that they will intersect, supposing a random chord to mean any line joining two points of the circle, all lengths less than the diameter being equally probable, and two chords drawn at random to mean two such chords inclined at any angle, all inclinations being equally probable.

General Solution by W. S. B. WOOLHOUSE, F.R.A.S.

The radius of the circle being taken as unity, let 2θ, $2\theta'$ denote the arcs of two chords, and to make the problem general, let $f\theta$ denote the function of which all values are equally probable, also let it be so stated that we may have $f(0) = 0$ and $f(\frac{1}{2}\pi) = 1$. Then, if we retain the values θ, θ', assume that $\theta' > \theta$, and suppose the chords to vary as to angular position, the probability that they shall intersect is evidently $\dfrac{2\theta}{\pi}$. Also, according to the hypothesis just stated, the proportion of cases of θ' having values greater than θ is $f(\frac{1}{2}\pi) - f(\theta) = 1 - f\theta$. The probability of intersection in respect of these cases is therefore $\dfrac{2\theta}{\pi}(1 - f\theta)$. Again, the proportion of cases of θ between θ and $\theta + d\theta$ is $df\theta$; and it is clear that the supplementary cases which have $\theta' < \theta$ must, when completed, involve just a repetition of the same combinations. Therefore, doubling the expression, the required probability is

$$p = \frac{4}{\pi}\int \theta\, df\theta\,(1 - f\theta) = \frac{4}{\pi}\left\{-\,\theta\,\frac{(1 - f\theta)^2}{2} + \int d\theta\,\frac{(1 - f\theta)^2}{2}\right\},$$

or, since the first term vanishes at both limits $\theta = 0$, $\theta = \frac{1}{2}\pi$,

$$p = \frac{2}{\pi}\int d\theta\,(1 - f\theta)^2,$$

which may be calculated for any stated form of the function $f(\theta)$.

In the particular case enunciated we have $f\theta = \sin\theta$, and

$$p = \frac{2}{\pi}\int d\theta\,(1-\sin\theta)^2 = \frac{2}{\pi}\left(\theta + 2\cos\theta - 2 + \frac{\theta - \cos\theta\sin\theta}{2}\right) = \frac{3\pi-8}{2\pi}.$$

If the chords be such that all distances from the centre of the circle are equally probable, then $f\theta = 1-\cos\theta$, and

$$p = \frac{2}{\pi}\int d\theta\cos^2\theta = \frac{\theta + \cos\theta\sin\theta}{\pi} = \frac{1}{2}.$$

If the chords be actually "drawn at random," $f\theta = \frac{2\theta}{\pi}$, and

$$p = \frac{2}{\pi}\int d\theta\left(1-\frac{2\theta}{\pi}\right)^2 = \frac{1}{3}\left\{1-\left(1-\frac{2\theta}{\pi}\right)^3\right\} = \frac{1}{3}.$$

Mr. WILSON, in his Solution to the case proposed (*Reprint*, Vol. VII., p. 20), arrives at one-half of the foregoing value, having omitted to include the combinations in which $l' > l$.

2270. (Proposed by Professor CAYLEY.)—To reduce the equation of a bicircular quartic into the form $SS'-k^3L = 0$, where $S = 0$, $S'=0$ are the equations of two circles, $L=0$ the equation of a line. (See Salmon's *Higher Plane Curves*, p. 128.)

Solution by the PROPOSER.

The equation of a bicircular quartic may be taken to be

$$(x^2+y^2)^2 + (u_1+u_0)(x^2+y^2) + v_2 + v_1 + v_0 = 0,$$

where, and in what follows, the subscript numbers denote the degrees in the coordinates (x, y) of the several functions to which they are attached.

Introducing an arbitrary constant θ_0, and putting the equation under the form

$$(x^2+y^2)^2 + (u_1+u_0-\theta_0)(x^2+y^2) + \theta_0(x^2+y^2) + v_2 + v_1 + v_0 = 0,$$

this may be identified with

$$(x^2+y^2+p_1+p_0)(x^2+y^2+q_1+q_0) + r_1 + r_0 = 0;$$

viz., the conditions in order to this identity are

$$p_1+p_0+q_1+q_0 = u_1+u_0-\theta_0,$$

$$(p_1+p_0)(q_1+q_0) + r_1 + r_0 = \theta_0(x^2+y^2) + v_2 + v_1 + v_0,$$

that is $\quad p_1+q_1 = u_1, \quad p_0+q_0 = u_0-\theta_0,$

$$p_1q_1 = \theta_0(x^2+y^2) + v_2, \quad p_1q_0 + p_0q_1 + r_1 = v_1, \quad p_0q_0 + r_0 = v_0.$$

Hence $\quad (p_1-q_1)^2 = u_1{}^2 - 4v_2 - 4\theta_0(x^2+y^2),$

where the right-hand side is a quadric function $(x, y)^2$, which, when the discriminant thereof is put $= 0$, (that is, when θ_0 is determined as the root of a quadric equation,) is a perfect square, p_1-q_1 is then a known linear function, and p_1+q_1 being equal to the linear function u_1, we have p_1 and q_1

as linear functions of (x, y). We may take for the constants p_0 and q_0 any values satisfying the equation $p_0+q_0 = u_0-\theta_0$; and we then have

$$r_1 = v_1-p_1q_0-p_0q_1, \quad r_0 = v_0-p_0q_0,$$

which completes the determination; the form

$$(x^2+y^2+p_1+p_0)\,(x^2+y^2+q_1+q_0)+r_1+r_0 = 0$$

is of course the same as the proposed form $SS'-k^3L = 0$.

Cor.—A somewhat more convenient form is $UU'-k^2V=0$, where $U=0$, $U'=0$ are the equations of two evanescent circles (pairs of imaginary lines), $V=0$ the equation of a circle; in fact the original form $SS'-k^3L = 0$ may be written $(S-\alpha)\,(S'-\alpha')+(\alpha S'+\alpha' S-\alpha\alpha'-k^3L) = 0$, which, when α, α' are so determined that $S-\alpha = 0$, $S'-\alpha'=0$ may be evanescent circles, is of the required form $UU'-k^2V=0$. The equation $UU'=0$ is that of the two pairs of tangents to the curve at the circular points at infinity respectively; in fact, writing $U=pq$, $U'=p'q'$, each of the lines $p=0$, $q=0$, $p'=0$, $q'=0$ meets the circle $V=0$ in one or other of the circular points at infinity, and therefore only in a single point not at infinity; hence each of these lines meets the curve $UU'-k^2V=0$ three times in one of the circular points at infinity, that is, the line in question is a tangent to one of the two branches through the circular point at infinity.

2273. (Proposed by M. Hermite.)—L'équation $1+ax+bx^2+\ldots = 0$ ayant toutes ses racines réelles, si l'on pose $(1+ax+bx^2+\ldots)^{-1} = 1+Ax+Bx^2+\ldots$, et qu'on opère avec la série ainsi obtenue sur une equation quelconque $F(x) = 0$, la nouvelle équation, savoir $F(x)+AF'(x)+BF''(x)+\ldots = 0$, aura au moins autant de racines imaginaires que la première.

Solution by Thomas Savage, M.A.

We see that $$Fx+AF'x+BF''x+\ldots = \left\{1+a\frac{d}{dx}+b\left(\frac{d}{dx}\right)^2+\ldots\right\}^{-1} Fx$$

$$= \frac{1}{\left(1-p\frac{d}{dx}\right)\left(1-q\frac{d}{dx}\right)\cdots} Fx,$$

where $\frac{1}{p}$, $\frac{1}{q}\ldots$ are the real roots of $1+ax+bx^2+\ldots\ldots = 0$.

Now let $\dfrac{1}{1-p\dfrac{d}{dx}} Fx = \phi x$. Then $\dfrac{d}{dx}\left(\epsilon^{-\frac{x}{p}}\phi x\right) = -\dfrac{1}{p}\epsilon^{-\frac{x}{p}} Fx$,

from which it appears that $\dfrac{d}{dx}\left(\epsilon^{-\frac{x}{p}}\phi x\right)$ has exactly as many real roots as Fx, and no more. But since $\dfrac{d}{dx}\left(\epsilon^{-\frac{x}{p}}\phi(x)\right)$ cannot become infinite, (ϕx

being a rational and integral function of x), $\frac{d}{dx}\left(\epsilon^{-\frac{x}{p}}\phi x\right)$ must vanish for some value of x between each adjacent pair of roots of $\epsilon^{-\frac{x}{p}}\phi(x)$, or the number of real roots of $\epsilon^{-\frac{x}{p}}\phi x$ can only exceed by one, at most, the number of real roots of $\frac{d}{dx}\left(\epsilon^{-\frac{x}{p}}\phi x\right)$, or of Fx. But ϕx and Fx are of the same degree. Hence the number of real roots of ϕx does not exceed the number of real roots of Fx, or in other words, the number of imaginary roots of ϕx is at least as great as those of Fx.

Having proved this for $\frac{1}{1-p\frac{d}{dx}}Fx$, the same method may be used to prove that $\frac{1}{\left(1-q\frac{d}{dx}\right)}\phi x = \frac{1}{\left(1-p\frac{d}{dx}\right)\left(1-q\frac{d}{dx}\right)}Fx$ has as many imaginary roots as ϕ, or as Fx. Proceeding thus we see that $Fx + AF'x + BF''x + \ldots.$ has at least as many imaginary roots as Fx.

2274. (Proposed by S. Roberts, M.A.).—If p_{kl} denote the perpendicular from K on the polar of L relative to a given conic, show that the equation in areal coordinates of the conic may be written

$$\frac{p_{aa}}{p_{oa}}x^2 + \frac{p_{bb}}{p_{ob}}y^2 + \frac{p_{cc}}{p_{oc}}z^2 + \frac{2p_{bc}}{p_{oc}}yz + \frac{2p_{ca}}{p_{oa}}zx + \frac{2p_{ab}}{p_{ob}}xy = 0,$$

the vertices of the triangle of reference and the centre being respectively A, B, C, O.

Solution by F. D. Thomson, M.A.

Let the equation in *areal* coordinates be

$$\phi(x, y, z) \equiv Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0 \ldots\ldots\ldots (1),$$

then the equation to the polar of first angular point of reference is

$$\frac{d\phi}{dx} \equiv Ax + Hy + Gz = 0;$$

therefore, if (x, y, z) be the coordinates of a point q, we have with the notation of the question

$$p_{qa} = \frac{Ax + Hy + Gz}{\left\{A, H, G\right\}},\ p_{qb} = \frac{Hx + By + Fz}{\left\{H, B, F\right\}},\ p_{qc} = \frac{Gx + Fy + Gz}{\left\{G, F, C\right\}},$$

when $\left\{A, H, G\right\}$ is a certain function of A, H, G.

But if q be the centre, we have

$$\frac{d\phi}{dx} = \frac{d\phi}{dy} = \frac{d\phi}{dz} = k \text{ suppose};$$

$$\therefore\ p_{oa} = \frac{k}{\left\{A, H, G\right\}},\ p_{ob} = \frac{k}{\left\{H, B, F\right\}},\ p_{oc} = \frac{k}{\left\{G, F, C\right\}}.$$

Again, $p_{aa} = \dfrac{A}{\left\{A, H, G\right\}},\ p_{bb} = \dfrac{B}{\left\{H, B, F\right\}},\ p_{cc} = \dfrac{C}{\left\{G, F, C\right\}}.$

$$p_{bc} = \frac{F}{\left\{G, F, C\right\}},\ p_{ca} = \frac{G}{\left\{A, H, G\right\}},\ p_{ab} = \frac{H}{\left\{H, B, F\right\}};$$

$$\therefore\ \frac{p_{aa}}{p_{oa}} = \frac{A}{k},\ \frac{p_{bb}}{p_{ob}} = \frac{B}{k},\ \frac{p_{cc}}{p_{oc}} = \frac{C}{k},\ \frac{p_{bc}}{p_{oc}} = \frac{F}{k},\ \frac{p_{ca}}{p_{oa}} = \frac{G}{k},\ \frac{p_{ab}}{p_{ob}} = \frac{H}{k};$$

hence, by substitution, the equation (1) takes the form given in the question.

2278. (Proposed by G. C. De Morgan, M.A.)—Supposing the probability of a point taken at hazard in space falling within the element of space dS to be $\left(\frac{c}{\pi}\right)^{\frac{3}{2}} \epsilon^{-cr^2} dS$, where r is the distance of dS from a fixed point O, and c some positive constant; determine the probability that the triangle formed by O, and two points taken at hazard in space, shall be acute-angled.

Solution by the Proposer.

Let OX, OY, OZ be three rectangular axes meeting in the point O. We may suppose one of the points to be taken at hazard in the positive part of the axis OX. Let P be any position of this point, OP $= u$. Let x, y, z be the coordinates of the other point. For a given position of P the probability (p) of an acute-angled triangle is

$$\left(\frac{c}{\pi}\right)^{\frac{3}{2}} \iiint \epsilon^{-c\,(x^2+y^2+z^2)}\, dx\, dy\, dz,$$

the integration being performed over the whole of the space lying outside the sphere described on OP as diameter, and between the plane YOZ and a plane parallel to it through P.

Let $y = r\cos\theta,\ z = r\sin\theta$; then we have

$$p = \left(\frac{c}{\pi}\right)^{\frac{3}{2}} \int_0^u \int_0^{2\pi} \int_{\sqrt{(ux-x^2)}}^{\infty} \epsilon^{-c\,(x^2+r^2)} r\,dx\, d\theta\, dr = \frac{1}{\sqrt{(\pi c)}}\ \frac{1}{u}\left(1 - \epsilon^{-cu^2}\right).$$

The probability that a point taken at hazard shall lie at a distance from O greater than u and less than $u + du$ (i.e. in the element of space contained between the surfaces of the spheres with centre O and radii u and $u + du$) is

$\frac{4c^{\frac{3}{2}}}{\sqrt{\pi}}\,\epsilon^{-cu^2}u^2\,du$. Multiplying this by p and integrating from 0 to ∞, we get $$\frac{4c}{\pi}\int_0^\infty\left(\epsilon^{-cu^2}-\epsilon^{-2cu^2}\right)u\,du=\frac{1}{\pi},$$ the probability required.

NOTE.—If we suppose c to diminish without limit, the law of probability given above approaches more and more nearly that of the case where all points are supposed to be distributed equally throughout space. It seems from this as if $\frac{1}{\pi}$ were the probability that three points taken at random in space should form an acute-angled triangle, since we are at liberty to suppose one point to be fixed in determining this probability.

It would be interesting to see the same problem solved for the case where all three points are taken at hazard, instead of one being fixed at the origin, if it could be done. I have been unable to perform the integrations, and fancy they would be complicated. I should like to see whether the probability when $c=0$ is the same in that case, but am inclined to believe that it would not be so.

2297. (Proposed by M. W. CROFTON, B.A.)—If two points are taken at random in a given line of length l, show that the probability of no one of the three segments exceeding a given length $c\,(>\frac{1}{3}l)$ is:

$$(1)\text{ if } c<\tfrac{1}{2}l,\ p_1=\left(\frac{3c-l}{l}\right)^2;\quad (2)\text{ if } c>\tfrac{1}{2}l,\ p_2=1-3\left(\frac{l-c}{l}\right).$$

Solution by STEPHEN WATSON.

Let AB be the line, C and D the points, and put AC $=x$, AD $=y$.

(1). In this case y must be $<x+c$ but $>b-c$, and the limits of x are $l-2c$ and c; hence multiplying by 2 because C and D may be interchanged, and dividing by l^2 the measure of the number of positions of the two points, we have

$$p_1=\frac{2}{l^2}\int_{l-2c}^{c}\int_{l-c}^{x+c}dx\,dy=\frac{2}{l^2}\int_{l-2c}^{c}(x+2c-l)\,dx=\left(\frac{3c-l}{l}\right)^2.$$

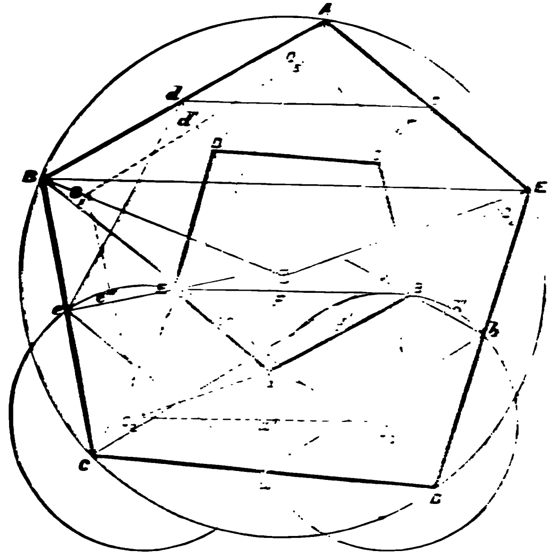
(2). In this case, the chances of each of the segments *exceeding* c must be equal; and in order that AC may exceed c the limits are y between x and l, and x from c to l;

hence $$p_2=1-3\int_c^l\frac{2}{l^2}(l-x)\,dx=1-3\left(\frac{l-c}{l}\right)^2.$$

[If we put $p_1=f(c)$, $p_2=\text{F}(c)$, we have the relation $f(c)-f(l-c)=\text{F}(c)-\text{F}(l-c)$. See the *Note* at the end of the solution to Quest. 1321, *Reprint*, Vol. IV., p. 86.]

2259. (Proposed by J. Griffiths, M.A.)—Prove *geometrically* that if A, B, C, D, E be five points on a circle, the consecutive intersections of the nine-point circles of the triangles ABC, BCD, CDE, DEA, EAB, lie on another circle whose radius is one-half that of the first.

Solution by James Dale.



Let O_2, O_3 be the centres of the nine-point circles of BCD, CDE; then, since the line joining the intersections of the perpendiculars of BCD, CDE is parallel and equal to BE, it follows that O_2O_3 is parallel to BE and equal to one half of BE. Let a, a'' be the middle points of CD, O_2O_3; join aa'', and make $a''A'$ equal to $a''a$, then A' is the second point of intersection of the nine-point circles of BCD, CDE. In the same manner may be constructed the points B', C', D', E'.

Join O_2A', O_2E', O_3A', O_3B'; then O_2O_3, $A'O_2$, $A'O_3$ are respectively equal to the halves of BE, OE, OB; and O_2O_3 is parallel to BE; $\therefore \angle O_2A'O_3 = BOE$, and consequently $\angle a''A'O_3 = BDE$. Again, the exterior angle $FA'B'$ (of the quadrilateral $A'abB'$) $= abB' = 90° - ACE$ (since bB' is perpendicular to AC) $= 90° - ABE$; also $\angle FA'E' = E'ea = 90° - ADB = 90° - AEB$; therefore $\angle E'A'B' = 180° - (ABE + AEB) = EAB$. In like manner the angles at B', C', D', E' may be proved equal to the angles at B, C, D, E; and the pentagon $A'B'C'D'E'$ being equiangular to ABCDE is also similar to it, and can be inscribed in a circle. Draw O_3d' perpendicular to $A'B'$; then $\angle a''O_3d' = FA'B' = 90° - ADE$, and $\angle A'O_3a'' = BEO = 90° - BDE$, therefore $\angle A'O_3d' = BDE - ADE = BDA$; and therefore since $A'B'$ subtends in the circle (O_3) the same angle that AB subtends in the circle (O), and the radius of (O_3) is one half that of (O), it follows that $A'B'$ is one half of AB, and that the circle circumscribing $A'B'C'D'E'$ is equal to any one of the nine-point circles (O_1), (O_2), &c. Moreover, $\angle B'O_3O_2 = 90^\circ + AEB - ABE$, and $\angle E'O_2O_3 = 90^\circ + ABE - AEB$, therefore $E'B'$ is parallel to O_2O_3, and consequently the sides of $A'B'C'D'E'$ are respectively parallel to the sides of $ABCDE$.

[An *analytical* Solution is given on p. 57 of Vol. V. of the *Reprint.*]

2303. (Proposed by A. RENSHAW.) — If $u = \log(x^3+y^3+z^3-3xyz)$, prove that

$$\Sigma . \left\{ \frac{d^n u}{dx^{\alpha+1} . dy^{\beta} . dz^{\gamma}} + \frac{d^n u}{dx^{\alpha} . dy^{\beta+1} . dz^{\gamma}} + \frac{d^n u}{dx^{\alpha} . dy \; . dz^{\gamma+1}} \right\}$$

$$= \frac{3n(n-1) . \underline{|n-1} . (-1)^{n-1}}{2(x+y+z)^n},$$

where α, β, γ are subject to the condition $\alpha+\beta+\gamma = n-1$, and have *each* of them all integral values from 0 to $(n-1)$ consistent with this condition; also Σ signifies the sum of all these values.

I. *Solution by the* PROPOSER.

It may easily be shown that

$$\frac{du}{dx} + \frac{du}{dy} + \frac{du}{dz} = \frac{3}{x+y+z} \quad \ldots\ldots\ldots\ldots\ldots\ldots (A);$$

and whether we take the $(n-1)$th differential coefficient of both sides of (A) entirely as regards only one of the variables, or partly as regards them all three, we shall evidently obtain the same result, viz.

$$\frac{3\,\underline{|n-1}\,(-1)^{n-1}}{(x+y+z)^n} = \frac{d^{n-1}}{dx^{n-1}}(A) = \frac{d^{n-1}}{dx^{\alpha} . dy^{\beta} . dz^{\gamma}}(A) \quad \ldots\ldots\ldots (B),$$

$\alpha+\beta+\gamma$ being equal to $n-1$.

Now the total number of variations of α, β, γ which satisfy the stated conditions will obviously be the sum of the series $1+2+\ldots+n$, that is $\frac{1}{2}n(n-1)$; and as each of these variations gives the same value for (B), we have the total sum, that is,

$$\Sigma . \left\{ \frac{d^n u}{dx^{\alpha+1} . \&c.} + \&c. \right\} = \frac{3n(n-1)\,\underline{|n-1}\,(-1)^{n-1}}{2(x+y+z)^n}.$$

II. *Solution by the* REV. J. L. KITCHIN, M.A.

We have $u = \log(x+y+z) + \log(x+\omega_1 y+\omega_2 z) + \log(x+\omega_2 y+\omega_1 z)$, where $1, \omega_1, \omega_2$ are the cube roots of unity, and therefore $\omega_1\omega_2 = 1$, and $1+\omega_1+\omega_2 = 0$.

Now $$\frac{du}{dx} = \frac{1}{x+y+z} + \frac{1}{x+\omega_1 y+\omega_2 z} + \frac{1}{x+\omega_2 y+\omega_1 z};$$

therefore $$\left(\frac{d}{dx}\right)^{\alpha} \left(\frac{d}{dy}\right)^{\beta} \left(\frac{d}{dz}\right)^{\gamma} \frac{du}{dx} = \left(\frac{d}{dx}\right)^{\alpha+1} \left(\frac{d}{dy}\right)^{\beta} \left(\frac{d}{dz}\right)^{\gamma} . u$$

$$=(-1)^{n-1}\,\underline{|n-1}\left\{\frac{1}{(x+y+z)^n} + \frac{\omega_1^{\beta}\,\omega_2^{\gamma}}{(x+\omega_1 y+\omega_2 z)^n} + \frac{\omega_2^{\beta}\,\omega_1^{\gamma}}{(x+\omega_2 y+\omega_1 z)^n}\right\}.$$

Similarly $$\left(\frac{d}{dx}\right)^{\alpha} \left(\frac{d}{dy}\right)^{\beta+1} \left(\frac{d}{dz}\right)^{\gamma}$$

$$=(-1^{n-1}\,\underline{|n-1}\left\{\frac{1}{(x+y+z)^n} + \frac{\omega_1^{\beta+1}\,\omega_2^{\gamma}}{(x+\omega_1 y+\omega_2 z)n} + \frac{\omega_2^{\beta+1}\,\omega_1^{\gamma}}{(x+\omega_2 y+\omega_1 z)^n}\right\},$$

and $$\left(\frac{d}{dx}\right)^{\alpha} \left(\frac{d}{dy}\right)^{\beta} \left(\frac{d}{dz}\right)^{\gamma+1}$$

$$=(-1)^{n-1}\,\underline{|n-1}\left\{\frac{1}{(x+y+z)^n} + \frac{\omega_1^{\beta}\,\omega_2^{\gamma+1}}{(x+\omega_1 y+\omega_2 z)^n} + \frac{\omega_2^{\beta}\,\omega_1^{\gamma+1}}{(x+\omega_2 y+\omega_1 z)^n}\right\},$$

therefore Sum

$$=(-1)^{n-1}\,\underline{|n-1}\left\{\frac{3}{(x+y+z)^n} + \frac{\omega_1^{\gamma}\,\omega_2^{\beta}(1+\omega_1+\omega_2)}{(x+\omega_1 y+\omega_2 z)^n} + \frac{\omega_2^{\beta}\omega_1(1+\omega_1+\omega_2)}{(x+\omega_2 y+\omega_1 z)^n}\right\}$$

$$=(-1)^{n-1}\,\underline{|n-1}\,\frac{3}{(x+y+z)^n}\ldots\ldots\ldots(\text{I}).$$

This result is quite independent of the values of α, β, γ, the only condition which they must fulfil being

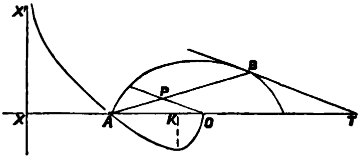
$$\alpha+\beta+\gamma = n-1\ldots\ldots\ldots\ldots(\text{II}).$$

Hence the *sum* in the question is (I) multiplied by the number of solutions of which (II) is susceptible. This number is easily seen to be given by the series, $1+2+3+\ldots\ldots+n-1 = \frac{1}{2}n(n-1)$,

therefore $$\Sigma\left(\frac{d^n u}{dx^{\alpha+1}\,\&c.} + \&c.\right) = \frac{(-1)^{n-1}\,\underline{|n-1}.3n(n-1)}{2(x+y+z)^n}.$$

2294. (Proposed by S. Watson.)—From one end (A) of the major axis of an ellipse draw any chord AB, and through the centre O draw OP parallel to the tangent at B, and meeting AB in P. Find the locus and area of the curve traced out by P.

Solution by R. TUCKER, M.A.; REV. J. L. KITCHIN, M.A.; H. TOMLINSON; *and others.*



Let AP $=\rho$, AB $=r$, $\angle$ BAO $=\theta$; and let the coordinates of B referred to O be (x, y); then if we draw the tangent BT we have, by similar triangles,

$$\frac{\rho}{r} = \frac{-a}{x}, \text{ and since}$$

$$r = \frac{2b^2}{a}\,\frac{\cos\theta}{1-e^2\cos^2\theta}$$

we get for locus of P the cubic

$$\frac{y^2}{x^2} = \frac{b^2}{a^2}\cdot\frac{a-x}{a+x} \quad\text{..(i.);}$$

therefore $$\frac{dy}{dx} = \pm\frac{b}{a}\cdot\frac{a^2-ax-x^2}{(a+x)\,(a^2-x^2)^{\frac{1}{2}}} \quad\text{..........................(ii.),}$$

and $$\frac{d^2y}{dx^2} = \mp\frac{ab\,(a-x)}{(a+x)\,(a^2-x^2)^{\frac{3}{2}}} \quad\text{..........................(iii.)}$$

Hence the curve passes through A and O and has an asymptote XX′(AX $=$ AO) perpendicular to AO, and the maximum ordinate corresponding to AK is found from (ii.) in the usual way. There is an exactly similar portion on the other side of AO. Let A be the area, and put $x = a\cos\theta$; then

$$A = 2\int y\,dx = 2ab\int\cos\theta\,(1-\cos\theta)\,d\theta = 2ab\,(\sin\theta - \tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta).$$

Hence the area of the loop $= 2ab\,(1-\frac{1}{4}\pi)$, and of the other portion $= 2ab\,(1+\frac{1}{4}\pi)$, whence the whole area $= 4ab =$ area of circumscribing rectangle.

2306. (Proposed by R. TUCKER, M.A.)—Eliminate θ between the following equations :—

$$x\cos(\theta-A)+y\cos\theta = 2R\sin(\theta+C)\cos\theta\cos(\theta-A)\ \text{......................(1),}$$

$$x\sin(\theta-A)+y\sin\theta = 2R\left\{\sin(\theta+C)\sin\theta\cos(\theta-A)-\cos B\cos\theta\right\}\text{..(2),}$$

where A, B, C are the angles of a plane triangle.

I. *Solution* by STEPHEN WATSON.

Eliminating first y and then x, we have

$$x\sin A = 2R\cos B\cos^2\theta,\quad y\sin A = 2R\cos C\cos^2(\theta-A),$$

hence, eliminating θ, the required result is

$$\cos^{-1}\left(\frac{x\sin A}{2R\cos B}\right)^{\frac{1}{2}} - \cos^{-1}\left(\frac{y\sin A}{2R\cos C}\right)^{\frac{1}{2}} = A,$$

or, $$\frac{x}{\cos B}+\frac{y}{\cos C}-2\left(\frac{xy}{\cos B\cos C}\right)^{\frac{1}{2}}\cos A = 2R\sin A,$$

or, $$\frac{x^2}{\cos^2 B}+\frac{y^2}{\cos^2 C}-\frac{2\cos 2A\,.\,xy}{\cos B\cos C}-4R\sin A\left(\frac{x}{\cos B}+\frac{y}{\cos C}\right)+4R^2\sin^2 A=0.$$

II. *Solution by the* REV. J. L. KITCHIN; E. MCCORMICK; *the* PROPOSER; *and many others.*

From $\{(1)\sin\theta-(2)\cos\theta\}$ and $\{(1)\cos\theta+(2)\sin\theta\}$ we have

$$x\sin A-R\cos B = R\cos 2\theta\cos B \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

$$x\cos A+y-R\sin B=R\left\{\cos 2\theta\sin(C-A)+2\cos A\cos C\sin 2\theta\right\}\ldots\ldots (4).$$

Substituting from (3) in (4), we get

$$-x\cos 2A\cos C+y\cos B-2R\sin A\cos B\cos C = 2R\cos A\cos B\cos C\sin 2\theta\ldots\ldots (5),$$

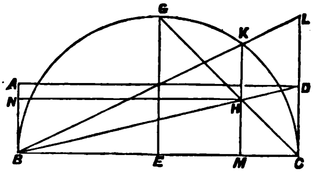
$$x\sin 2A\cos C-2R\cos A\cos B\cos C = 2R\cos A\cos B\cos C\cos 2\theta\ldots (6);$$

hence, from $(5)^2+(6)^2$ we obtain finally,

$$x^2\sec^2 B+y^2\sec^2 C-2xy\cos 2A\sec B\sec C-4R\sin A\,(x\sec B+y\sec C)+4R^2\sin^2 A = 0.$$

2000. (Proposed by C. M. INGLEBY, LL.D.)—Let ABCD be a rectangle. Bisect one of the sides, BC, in E, and with centre E and distance EB describe a circle. From E draw the radius EG (towards AD) parallel to CD, and join GC. Then join BD, cutting GC (or, as the case may be, GC produced) in H. From H draw the straight line HK, parallel to EG, meeting the circle (towards AD) in K. Join BK and produce it to meet CD (or CD produced) in L: then CL shall be the side of a square equal to the rectangle ABCD.

Solution by C. LAW; W. M. MADDEN; H. TOMLINSON; J. DALE; W. H. LAVERTY; *the* PROPOSER; *and others.*



Produce KH to meet BC in M, and through H draw a line parallel to BC meeting AB in N. Then, since HM is equal to CM, the rectangle NM is equal to the square on KM. But the rectangle NM being similar to AC, we have

$$NM : AC = HM^2 : DC^2 = KM^2 : LC^2;$$

and $NM = KM^2$, therefore $AC = LC^2$.

2236. (Proposed by the Rev. J. BLISSARD.)—Prove that

$$(1.)\quad \tan^{-1}x = \frac{x}{\sqrt{(1+x^2)}}\left\{1+\frac{1}{2}\cdot\frac{1}{3}\left(\frac{x^2}{1+x^2}\right)+\frac{1.3}{2.4}\cdot\frac{1}{5}\left(\frac{x^2}{1+x^2}\right)^2+\&c.\right\};$$

$$(2.)\quad \pi = \frac{28}{10}\left\{1+\frac{2}{3}\cdot\frac{2}{10^2}+\frac{2.4}{3.5}\cdot\left(\frac{2}{10^2}\right)^2+\frac{2.4.6}{3.5.7}\cdot\left(\frac{2}{10^2}\right)^3+\&c.\right\}$$
$$+\frac{30336}{10^5}\left\{1+\frac{2}{3}\cdot\frac{12^2}{10^5}+\frac{2.4}{3.5}\cdot\left(\frac{12^2}{10^5}\right)^2+\frac{2.4.6}{3.5.7}\cdot\left(\frac{12^2}{10^5}\right)^3+\&c.\right\}.$$

Solution by the PROPOSER.

By the aid of what I have called *Representative Notation*, we readily obtain the following highly general formula, (see *Quarterly Journal of Mathematics*, No. 22, p. 173,) viz.

$$\left(\frac{x}{1+x}\right)^n+\frac{n}{1}\cdot\frac{m}{m+r}\left(\frac{x}{1+x}\right)^{n+1}+\frac{n(n+1)}{1.2}\cdot\frac{m(m+1)}{(m+r)(m+r+1)}\left(\frac{x}{1+x}\right)^{n+2}$$
$$+\ldots = x^n-\frac{n}{1}\cdot\frac{r}{m+r}x^{n+1}+\frac{n(n+1)}{1.2}\cdot\frac{r(r+1)}{(m+r)(m+r+1)}x^{n+2}-\ldots$$

(1.) Let $m=\frac{1}{2}$, $n=\frac{1}{2}$, $r=1$; then, putting x^2 for x, we have

$$\frac{x}{\sqrt{(1+x^2)}}\left\{1+\frac{1}{2}\cdot\frac{1}{3}\left(\frac{x^2}{1+x^2}\right)+\frac{1.3}{2.4}\cdot\frac{1}{5}\left(\frac{x^2}{1+x^2}\right)^2+\ldots\right.$$
$$= x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots = \tan^{-1}x\,\ldots\ldots\ldots\ldots\ldots\ldots(\text{I}).$$

Let $x=\frac{1}{\sqrt{3}}$; then $\tan^{-1}\frac{1}{\sqrt{3}}=\frac{\pi}{6}$, and

$$\pi = 3\left(1+\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{4}+\frac{1.3}{2.4}\cdot\frac{1}{5}\cdot\frac{1}{4^2}+\frac{1.3.5}{2.4.6}\cdot\frac{1}{7}\cdot\frac{1}{4^3}+\ldots\right).$$

This series for the evaluation of π is probably new, and the most highly converging *single* series for that purpose which has yet been discovered.

(2.) In the above formula, let $m=1$, $n=1$, $r=\frac{1}{2}$; then, putting x^2 for x,

$$\frac{x^2}{1+x^2}+\frac{2}{3}\left(\frac{x^2}{1+x^2}\right)^2+\frac{2.4}{3.5}\left(\frac{x^2}{1+x^2}\right)^3+\ldots = x^2-\frac{x^4}{3}+\frac{x^6}{5}-\ldots = x\tan^{-1}x.$$

therefore $\tan^{-1}x = \frac{x}{1+x^2}\left\{1+\frac{2}{3}\left(\frac{x^2}{1+x^2}\right)+\frac{2.4}{3.5}\left(\frac{x^2}{1+x^2}\right)^2+\ldots\right\}$,

a known formula, which has been independently discovered by various writers.

In this formula put $\frac{a}{b}$ for x; then we have

$$\tan^{-1}\frac{a}{b} = \frac{ab}{a^2+b^2}\left\{1+\frac{2}{3}\left(\frac{a^2}{a^2+b^2}\right)+\frac{2.4}{3.5}\left(\frac{a^2}{a^2+b^2}\right)^2+\ldots\right\}\ldots\ldots(\text{II.})$$

Now, in order that the right-hand series may be most readily calculable, it is requisite that $\frac{a^2}{a^2+b^2}$ should be of the form $\frac{m}{10^n}$, which is accomplished if a^2+b^2 is of the form $2^p.5^q$.

Let $\tan^{-1}\frac{e}{f} = \tan^{-1}\frac{a}{b} - \tan^{-1}\frac{c}{d} = \tan^{-1}\frac{ad-bc}{ac+bd}$,

then $\frac{e}{f} = \frac{ad-bc}{ac+bd}$, and $\frac{e^2}{e^2+f^2} = \frac{(ad-bc)^2}{(a^2+b^2)(c^2+d^2)}$:

and if both a^2+b^2, c^2+d^2 are of the form $2^p.5^q$, $\frac{e^2}{e^2+f^2}$ will be of the required form, viz. $\frac{m}{10^n}$. We apply this as follows:—

$$\tan^{-1}1 - \tan^{-1}\frac{1}{2} = \tan^{-1}\frac{1}{3}, \quad \tan^{-1}\frac{1}{2} - \tan^{-1}\frac{1}{3} = \tan^{-1}\frac{1}{7},$$

$$\tan^{-1}\frac{1}{3} - \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{2}{11}, \quad \tan^{-1}\frac{2}{11} - \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{3}{79},$$

$$\tan^{-1}\frac{1}{7} - \tan^{-1}\frac{3}{79} = \tan^{-1}\frac{29}{278}, \quad \tan^{-1}\frac{29}{278} - \tan^{-1}\frac{3}{79} = \tan^{-1}\frac{1457}{22049},$$

$$\tan^{-1}\frac{1457}{22049} - \tan^{-1}\frac{3}{79} = \tan^{-1}\frac{24478}{873121},$$

$$\tan^{-1}\frac{3}{79} - \tan^{-1}\frac{24478}{873121} = \tan^{-1}\frac{685601}{69049993}, \text{ \&c.}$$

Here all the quantities $\frac{1}{2}, \frac{1}{3}, \frac{1}{7}, \frac{2}{11}, \frac{3}{79}, \ldots \frac{685601}{69049993}$, possess the property that if $\frac{a}{b}$ be put for any one of them, $\frac{a^2}{a^2+b^2}$ is of form $\frac{m}{10^n}$.

Hence $\pi = 4\tan^{-1}\frac{1}{2} + 4\tan^{-1}\frac{1}{3} = 8\tan^{-1}\frac{1}{3} + 4\tan^{-1}\frac{1}{7}$

$$= 8\tan^{-1}\frac{2}{11} + 12\tan^{-1}\frac{1}{7} = 20\tan^{-1}\frac{1}{7} + 8\tan^{-1}\frac{3}{79}$$

$$= 20\tan^{-1}\frac{29}{278} + 28\tan^{-1}\frac{3}{79} = 20\tan^{-1}\frac{1457}{22049} + 48\tan^{-1}\frac{3}{79}$$

$$= 68\tan^{-1}\frac{3}{79} + 20\tan^{-1}\frac{24478}{873121} = 88\tan^{-1}\frac{24478}{873121} + 68\tan^{-1}\frac{685601}{69049993} \text{ \&c.}$$

From the second of these expressions we obtain, by (II.),

$$\pi = \frac{24}{10}\left(1 + \frac{2}{3}\cdot\frac{1}{10} + \frac{2.4}{3.5}\cdot\frac{1}{10^2} + \ldots\right) + \frac{56}{10^2}\left(1 + \frac{2}{3}\cdot\frac{2}{10} + \frac{2.4}{3.5}\cdot\frac{2^2}{10^4} + \ldots\right);$$

from the fourth, $\pi = \frac{28}{10}\left(1 + \frac{2}{3}\cdot\frac{2}{10^2} + \frac{2.4}{3.5}\cdot\frac{2^2}{10^4} + \ldots\right)$

$$+ \frac{30336}{10^5}\left(1 + \frac{2}{3}\cdot\frac{12^2}{10^5} + \frac{2.4}{3.5}\cdot\frac{12^4}{10^{10}} + \ldots\right);$$

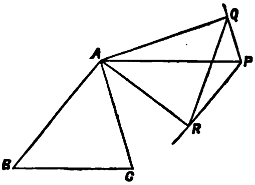
and from the last

$$\pi = \frac{88.24478.873128.2^{17}}{10^{17}}\left(1 + \frac{2}{3}\cdot\frac{(12239)^2.2^{18}}{10^{17}} + \frac{2.4}{3.5}\cdot\frac{(12239)^4.2^{36}}{10^{34}} + \ldots\right)$$

$$+ \frac{68.2^{21}.685601.69049993}{10^{22}}\left(1 + \frac{2}{3}\cdot\frac{(685601)^2.2^{21}}{10^{22}} + \frac{2.4}{3.5}\cdot\frac{(685601)^4.2^{42}}{10^{44}} + \ldots\right).$$

2329. (Proposed by A. Renshaw.)—Let ABC be a triangle; on the *same* side of AB as C draw RP parallel to AB, and at a distance from it equal to AC; also on the *opposite* side of AC to B, draw PQ parallel to AC, and at a distance from it equal to AB; join AP, then AP is parallel to BC.

Solution by S. Watson; H. Murphy; E. McCormick;
the Proposer; *and many others.*



Let AR, AQ be the perpendiculars from A upon PR, PQ; and join RQ. Then obviously the triangle AQR is equal to ABC in all respects. Also a circle will pass through A, Q, P, R; hence

$$\angle \text{APR} = \text{AQR} = \text{ABC},$$

therefore AP and BC are parallel.

Corollary.—The trilinear equation of the line AP is plainly $b\beta + c\gamma = 0$, and the equation $a\alpha + b\beta + c\gamma = 0$ is evidently that of a line passing through the point of intersection of $b\beta + c\gamma = 0$ and $\alpha = 0$, which point is at infinity, since AP has been shown to be parallel to BC. Similarly, $a\alpha + b\beta + c\gamma = 0$ may be shown to be the equation of a line through the intersection of the other sides and parallels to them through the opposite angles, which points are therefore all at infinity; hence the line $a\alpha + b\beta + c\gamma = 0$ is altogether at infinity.

2238. (Proposed by R. Tucker, M.A.)—Find (1) the envelope, (2) the locus of centres, of circles cutting a given circle orthogonally, and intercepting a given length on a given straight line; also find (3) the envelope of the radical axis of the given and the orthogonal circles.

Solution by W. H. Laverty; E. McCormick; *the* Proposer;
and others.

1. Let $x^2 + y^2 = a^2$ be the equation of the given circle, and $x = c$ that of the given line on which a length $2b$ is intercepted by each of the orthogonal circles; then if $(x - x')^2 + (y - y')^2 = r'^2$ be the equation of one of these circles, we have for the given conditions

$$x'^2 + y'^2 = a^2 + r'^2, \text{ and } b^2 = r'^2 - (c - x')^2,$$

whence, eliminating r', we have for the circle and condition

$$x^2 + y^2 - 2xx' - 2yy' + a^2 = 0, \text{ and } b^2 = y'^2 + 2cx' - a^2 - c^2;$$

whence $$xdx' + ydy' = 0, \text{ and } cdx' + y'dy' = 0;$$

therefore $$y' = \frac{cy}{x}, \text{ and } x' = \frac{(a^2 + b^2 + c^2)\,x^2 - c^2y^2}{2cx^2};$$

hence the envelope required is the cubic curve

$$cx(x^2+y^2+a^2) = c^2y^2+(a^2+b^2+c^2)x^2$$

which has the centre of the given circle for a conjugate point, and the fixed line for an asymptote.

2. From what is given above the locus of centres is the parabola

$$y^2+2cx = a^2+b^2+c^2.$$

3. The equation of the radical axis is $xx'+yy = a^2$, and its envelope is readily found to be

$$(a^2+b^2+c^2)x^2+c^2y^2 = 2a^2cx,$$

an ellipse passing through the centre of the given circle.

AT RANDOM. BY HUGH GODFRAY, M.A.

In a former paper (*Reprint*, Vol. VI., p. 72) I attempted to show that the discordant results arrived at in some problems in the theory of chances were probably due to a want of definiteness in the word "random," which occurs in the statement of the question, and I spoke of a "random line drawn according to a special law." Mr. WILSON, in some remarks on my paper (Vol. VI., p. 82), objects to this expression as being unintelligible, and says that a line drawn at random must be a line drawn according to no law. I might point out that, in the very paragraph which precedes this objection, he has himself urged the necessity (and I fully agree with him) of defining a random point; and he gives as a "special law for a random point," that its chance of falling on any area is proportional to that area. If a random point must be thus defined, why not a random line? I hold that the necessity is even greater,—because, although there would be no difficulty in assigning other laws for a random point besides that just given, this one has the advantage of simplicity, and is probably that which many persons would consider most natural. But, with a random line, the matter is not quite so obvious and simple.

Let us consider a limited area—say a circle—across which a random line has to be drawn.

Firstly: We may define a random line to be a line joining two random points of the area,—a random point being, as above, one whose chance of falling on any area is proportional to that area.

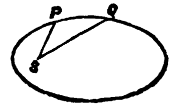
Secondly: We may define a random line to be one that crosses the circumference in two random points,—a random point on the circumference being one whose chance of falling on any arc is proportional to the length of that arc.

Here then we have two essentially different laws for a random line across the circle, and I shall not presume to settle which is the more natural or simple; but there can be no question that the result of a problem in which such a random line would enter, would generally be affected by the special law adopted.

With the first law the chance of the line's being a given diameter would be infinitely greater than that of its being a given tangent; with the second law, these chances would be equal.

I admit that in some problems one meaning may seem to attach itself to the term "random" more naturally than any other; that is, most persons would assume the same law. This would possibly be the case in the problem of "two chords drawn at random in a circle," to which I refer in the same Paper (p. 72), and on which Mr. Woolhouse has made a few remarks (p. 81). His solution is based on the second definition of a random chord given above; and is perhaps, as he expresses it, "consistent with our elementary notions of a random chord," but I do not therefore allow that it is the "only legitimate solution." Every solution is legitimate which is consistent with the data; and if several solutions arise from the ambiguity of the question, the fault is in the question, not in the solution.

As another illustration, I would offer the following problem. A line is drawn at random from the focus of an ellipse to the perimeter, what is the chance that its length is greater than the semi-major axis?



Let SP, SQ be any two focal lines. Then, we may define a random radius to be

Firstly: A line whose chance of falling within the angle PSQ is proportional to the angle PSQ.

The chance obtained from this will be $\frac{\cos^{-1} e}{\pi}$.

Secondly: A line whose chance of falling on the arc PQ is proportional to the length of the arc PQ.

This gives the chance $\frac{1}{2}$.

Thirdly: A line whose chance of falling on the area PSQ is proportional to that area.

The chance will now be $\frac{\pi+2e}{2\pi}$.

Here are three laws for a random line leading to three different results, and I should be puzzled to say which of the three solutions is the most natural.

I shall not multiply examples, but I trust that what I have said will explain what I meant when speaking of a random line drawn according to a special law.

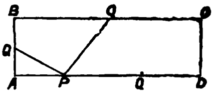
I should wish to add a few words on Mr. Wilson's solution of the four-point problem. In the remarks which I made (p. 73), I stated that he had obtained a correct result provided we adopt as the definition of a random point that it is the intersection of two random lines, and that this is irrespective of the law which regulates these random lines.

Now if we reverse this process, and define a random line to be a line joining two random points, these points being taken as in his own definition (p. 82); then it might at first sight seem as if his substitution of four lines for four points were perfectly legitimate: thus,—that if four random points A, B, C, D be taken, and these be joined by four lines AB, CD, AC, BD, we have a quadrilateral formed by four random lines. But this would be a mistake; for, although one pair of lines would be correctly drawn at random by joining the random points A, B and C, D, the other pair obtained by joining A, C and B, D would no longer be random lines. To complete the construction, two more pairs of random points would be required, making eight random points in all, two for each line. The conditions are therefore not the same.

Nor do I see that by increasing the number of random lines indefinitely, we obtain, as Mr. Wilson supposes, points uniformly distributed throughout space. For if three of the random lines cut a fourth in points P, Q, R such that the ratio of PQ to QR is incommensurable, no subsequent multiplication of the number of lines can ever divide these two distances in a uniform manner.

1860. (Proposed by M. W. Crofton, B.A.)—Two points being taken at random on the *perimeter* of a rectangle, find the chance of the distance between them being less than a given length.

I. *Solution by* Stephen Watson.



Let ABCD be the rectangle, and put $AD = a$, $AB = b$, $c^2 - a^2 = a_1^2$, $c^2 - b^2 = b_1^2$, where c is the given distance. We will first find the number of ways in which the points can lie within the distance c of each other, in the following cases.

1. When they both lie on the same side, length l, the number is

$$2\left\{\int_0^{l-c} c + \int_{l-c}^{l} (l-x)\right\} dx = c\,(2l-c) \;\ldots\ldots\ldots\ldots (1),$$

and this for $c = l$, or any greater value of c, must $= l^2$ ....(2).

2. When the points lie on contiguous sides, the number is

$$2\left\{\int_0^{b_1} b + \int_{b_1}^{a} \sqrt{(c^2 - x^2)}\right\} dx = c^2\left(\sin^{-1}\frac{a}{c} - \sin^{-1}\frac{b_1}{c}\right) + aa_1 + bb_1 \ldots (3),$$

when $c > a$. When $c < a$, and then $< b$, this becomes respectively

$$c^2\left(\frac{\pi}{2} - \sin^{-1}\frac{b_1}{c}\right) + bb_1 \text{ and } c^2\left(\frac{\pi}{2}\right) \;\ldots\ldots\ldots\ldots\ldots (4).$$

3. When the points lie on opposite sides, AD and BC, the number (according as b_1 is $<$ or $>$ $\frac{1}{2}a$) is

$$4\left\{\int_0^{b_1} (x + b_1) + \int_{b_1}^{\frac{1}{2}a} 2b_1\right\} dx \quad \text{or} \quad 4\left\{\int_0^{a-b_1} (x + b_1) + \int_{a-b_1}^{\frac{1}{2}a} a\right\} dx \ldots (5),$$

which give the same value $4b_1\,(a - \frac{1}{2}b_1)$. Similarly when the points lie on AB and CD, the number is $4a_1\,(b - \frac{1}{2}a_1)$......................(6).

In applying these, (1) and (2) must be doubled on account of the pairs of equal sides, and (3) and (4) twice doubled because of the four pairs of contiguous sides.

Now the whole number of ways the points can be taken on the sides is $4(a+b)^2$; hence, if p_1, p_2, p_3 be the chances when $c < b$; $c > b$ but $< a$;

$c > a$ but $< \surd(a^2 + b^2)$; in order to find p_1 we must take (1), first when $l = a$, then when $= b$, and the last in (4); therefore

$$p_1 = \frac{c}{a+b} + \frac{c^2}{(a+b)^2}\left(\frac{\pi}{2} - 1\right).$$

Similarly, from (1), (2), the first of (4) and (5), we have

$$p_2 = \frac{b_1}{a+b} + \frac{c^2}{(a+b)^2}\left(\sin^{-1}\frac{b}{c} + \frac{ac + b^2 - c^2}{c^2}\right).$$

Lastly, from (2), (3), (5), and (6), we have

$$p_3 = \frac{a_1 + b_1}{a+b} + \frac{c^2}{(a+b)^2}\left(\sin^{-1}\frac{a}{c} - \sin^{-1}\frac{b_1}{c} + \frac{a^2 + b^2 - c^2}{c^2}\right).$$

II. *Solution by the* PROPOSER.

I. Using the notation and figure of the foregoing Solution, let c be less than a or b. Calling the points P, Q, the favourable cases will be:—

(1.) When P, Q are on adjacent sides, *e.g.*, P on AD, Q on AB; if AP $= x$, AQ $= y$, we have $F = \iint dx\,dy$, the limits being given by $x^2 + y^2 < c^2$, $x > 0$, $y > 0$; that is, the quarter of a circle of radius c, or $\frac{1}{4}\pi c^2$. Now 8 times this gives all the favourable cases in this category; hence $F = 2\pi c^2$.

(2.) When P, Q are both on one side, as AD. Taking first the cases where P is nearer to A than Q is, and putting AP $= x$ and DQ $= y$, the favourable cases are measured by $\iint dx\,dy$, the limits being $x + y < a$, $x + y > a - c$, $x > 0$, $y > 0$, that is $ac - \frac{1}{2}c^2$. Doubling this for the cases when Q is nearest to A, and doubling again for the side BC; and likewise for AB, CD; we have $F = 4ac + 4bc - 4c^2$. Hence the total number of favourable cases is

$$F = 2\pi c^2 + 4(a+b)c - 4c^2.$$

As the whole number of cases is represented by $4(a+b)^2$, the required probability is

$$p_1 = \frac{c}{a+b} + \frac{c^2}{(a+b)^2}\left(\frac{\pi}{2} - 1\right).$$

II. When $a > c > b$. (1.) Let P be on AD, Q on AB; the limits of $\iint dx\,dy$ are $\quad x^2 + y^2 < c^2$, $y < b$, $x > 0$, $y > 0$.

It is easily seen, by referring to the geometrical area which the integral represents, that

$$\iint dx\,dy = \tfrac{1}{2}bb_1 + \tfrac{1}{2}c^2\sin^{-1}\frac{b}{c}; \quad \therefore\ F = 4bb_1 + 4c^2\sin^{-1}\frac{b}{c}.$$

(2.) When both points are on AD or BC, $F = 4ac - 2c^2$.

(3.) When both are on AB or CD, $F = 2b^2$.

(4.) If P is on AD, and Q on BC. Put $x = \text{AP}$, $y = \text{BQ}$; confining our attention to the cases where $x > y$, we have for the limits of $\iint dx\,dy$, since $b^2 + (x-y)^2 < c^2$,

$$x - y < b_1, \quad a > x > y > 0; \quad \therefore \iint dx\,dy = ab_1 - \tfrac{1}{2}b_1^2.$$

Doubling this, and doubling again for the cases where P is in BC, Q in AD, we have

$$F = 4ab_1 - 2b_1^2.$$

Adding all these values of F, we have, for the probability in this case,

$$p_2 = \frac{b_1}{a+b} + \frac{c^2}{(a+b)^2}\left(\sin^{-1}\frac{b}{c} - 1 + \frac{a}{c} + \frac{b^2}{c^2}\right).$$

III. When $c > a > b$. (1) When P, Q are on adjacent sides, $F = 8\iint dx\,dy$, the limits being $x^2+y^2<c^2,\ y<b,\ x<a,\ y>0,\ x>0$;

whence
$$F = 4c^2\left(\sin^{-1}\frac{a}{c} + \sin^{-1}\frac{b}{c} - \frac{\pi}{2}\right) + 4aa_1 + 4bb_1.$$

(2.) If P, Q are both on one side, $F = 2a^2 + 2b^2$.

(3.) If P, Q are on opposite sides, we have, as in (II.),

$$F = 4aa_1 + 4bb_1 - 4c^2 + 2a^2 + 2b^2.$$

$$\therefore\ p_3 = \frac{a_1+b_1}{a+b} + \frac{c^2}{(a+b)^2}\left(\sin^{-1}\frac{a}{c} + \sin^{-1}\frac{b}{c} - \frac{\pi}{2} - 1 + \frac{a^2+b^2}{c^2}\right).$$

If $F(a, b, c)$, $f(a, b, c)$, $\phi(a, b, c)$ be these three functions, giving the the probability in each case, we find the following remarkable relation between their forms,

$$F(a, b, c) + \phi(a, b, c) = f(a, b, c) + f(b, a, c).$$

[See *Reprint*, Vol. IV., p. 86, and Vol. V., p. 53.]

1874. (Proposed by W. S. BURNSIDE, M.A.)—To apply PLUCKER'S Theory to determine the foci of a Cartesian oval.

Solution by the PROPOSER.

1. Let its equation be $l\left\{(x-c)^2+y^2\right\}^{\frac{1}{2}} + m\left\{(x+c)^2+y^2\right\}^{\frac{1}{2}} = d$; then, clearing of radicals, and eliminating y by the equation $x+iy=p$ (where $i=\sqrt{-1}$), also writing for shortness

$$l^2+m^2=\alpha,\ \ l^2-m^2=\beta,\ \ p\beta-c\alpha=q,\ \ p\alpha-c\beta=q_1,\ \ \gamma^2=c^2-p^2,$$

we find
$$d^4 - 2d^2(\gamma^2\alpha+2q_1x) + (\gamma^2\beta+2qx)^2 = 0,$$

or
$$4q^2x^2 - 4(q_1d^2-\gamma^2\beta q)x + \beta^2\gamma^4 - 2\alpha\gamma^2d^2 + d^4 = 0 \ldots\ldots\ldots (A).$$

If this equation have equal roots, or if the line $x+iy=p$ touch the curve, we have

$$\gamma^2\left\{d^2 - 2c(c\alpha - p\beta)\right\} = 0,$$

or
$$p \pm c = 0, \text{ and } d^2 = 2c\left\{(c-p)l^2 + (c+p)m^2\right\},$$

giving the positions of three of the foci. But as (A) is an equation of the second degree only, the circular points at infinity are double points on the curve, and if $q=0$ the line $x+iy=p$ meets the curve in three consecutive points at infinity, that is to say, it is a tangent at one of these double points; also, as $q=0$ gives only one value for p, there is only one tangent at this double point, that is to say, it is a cusp. The triple focus is hence given by $q=0$, or $(p-c)\,l^2=(p+c)\,m^2$. (See Salmon's *Higher Plane Curves*, p. 123.)

2. The following method may be sometimes convenient, when we know beforehand on what locus the focus we seek lies. Taking the present case, for example, and assuming that a third focus C lies in a straight line with the other two, A and B, and writing the equation of the Cartesian oval under the form $l\mathrm{A}^{\frac{1}{2}}+m\mathrm{B}^{\frac{1}{2}}=d$ (Salmon's *Higher Curves*, pp. 123 and 124), we have also $bc\,.\,\mathrm{A}+ca\,.\,\mathrm{B}+ab\,.\,bc\,.\,ca\equiv ba\,.\,\mathrm{C}$, whence, if C be a focus of the curve, $bc\,.\,\mathrm{A}+ca\,.\,\mathrm{B}+ab\,.\,bc\,.\,ca=0$ must satisfy the condition of touching $l\mathrm{A}^{\frac{1}{2}}+m\mathrm{B}^{\frac{1}{2}}=d$. And this is easily obtained, as we may consider A and B for the present as ordinary coordinates, and if $bc\,.\,\mathrm{A}+ca\,.\,\mathrm{B}+ab\,.\,bc\,.\,ca=0$ as a line touch $l\mathrm{A}^{\frac{1}{2}}+m\mathrm{B}^{\frac{1}{2}}=0$ as a conic we have $ab\,(l^2ac-m^2bc)=d^2$ to determine the point C.

1889. (Proposed by Chief Justice COCKLE, F.R.S.)—Solve, finitely, the differential equation $\frac{dy}{dx}+by^2=\frac{a}{x^6}$, or differential together with multiple of square of dependent variable equal to multiple of inverse sixth power of independent variable.

Solution by the PROPOSER.

Unless "finitely" be interpreted "by definite integrals," I am disappointed in my Solution. If it be so interpreted, then a combination of the results given in my Solutions of Questions 1854 and 1905 (*Reprint*, Vol. V., p. 50, and Vol. VI., p. 33) leads to a solution, and relations between the definite integrals which enter respectively into the solutions of $f(-6)=0$ and $f(+2)=0$ may be established. The three Questions are connected with the equations of RICCATI, on which I seize this opportunity to make some remarks.

It has been shown (see Solution of Quest. 1905) that if $\frac{d^2y}{dx^2}+ax^my=f(m)$, then the Solution of $f(m)=0$ may be made to depend upon that of $f\left(-\frac{m}{m+1}\right)=0$ and also upon that of $f\left(-\frac{3m+4}{m+1}\right)=0$ and $f(-m-4)=0$.

Let
$$m=-\frac{4n}{2n\pm1}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\;(1),$$
then
$$-\frac{m}{m+1}=-\frac{4n}{2n\mp1},\quad -\frac{3m+4}{m+1}=-\frac{4(n\mp1)}{2(n\mp1)\pm1},\quad -m-4=-\frac{4(n\pm1)}{2(n\pm1)\mp1}$$
$$\ldots\ldots\;(2,\,3,\,4),$$
consequently these transformations do not enable us to pass from a case in which n is an integer to a case in which n is a fraction.

The case $f(-1)=0$ is peculiar. If we multiply

$$\frac{d^2y}{dx^2}+\frac{a}{x}.y=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

into x, differentiate the product n times, and divide the result by x, we are conducted to an equation of the form

$$\frac{d^2y}{dx^2}+\frac{n}{x}\cdot\frac{dy}{dx}+\frac{a}{x}.y=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Change the independent variable from x to t, and (6) becomes

$$\frac{d^2y}{dt^2}+\left(\frac{n}{x}\cdot\frac{dx}{dt}-\frac{dt}{dx}\cdot\frac{d^2x}{dt^2}\right)\frac{dy}{dt}+\frac{a}{x}\left(\frac{dx}{dt}\right)^2y=0 \quad \ldots\ldots (7).$$

Let
$$\frac{n}{x}\cdot\frac{dx}{dt}-\frac{dt}{dx}\cdot\frac{d^2x}{dt^2}=\frac{\lambda}{t} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8),$$

then
$$\frac{1}{x^n}\cdot\frac{dx}{dt}=\frac{\mathrm{C}}{t^\lambda}, \qquad x=\left(\frac{\mathrm{C}(1-n)}{1-\lambda}\right)^{\frac{1}{1-n}}.\, t^{\frac{1-\lambda}{1-n}} \quad \ldots\ldots\ldots (9,10),$$

$$\frac{dx}{dt}=\mathrm{C}^{\frac{1}{1-n}}.\left(\frac{1-n}{1-\lambda}\right)^{\frac{n}{1-n}}.\, t^{\frac{n-\lambda}{1-n}}, \quad \frac{a}{x}\left(\frac{dx}{dt}\right)^2=a\mathrm{C}^{\frac{1}{1-n}}.\left(\frac{1-n}{1-\lambda}\right)^{\frac{2n-1}{1-n}}.\, t^{\frac{2n-\lambda-1}{1-n}}$$
$$\ldots\ldots\ldots (11, 12).$$

Hence we may transform (6) or (7) into (B being free from x and t)

$$\frac{d^2y}{dt^2}+\frac{\lambda}{t}\cdot\frac{dy}{dt}+\mathrm{B}t^{\frac{2n-\lambda-1}{1-n}}.y=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (13),$$

and, giving to λ the values 0 and 2 successively, (13) becomes successively

$$\frac{d^2y}{dt^2}+\mathrm{B}t^{\frac{2n-1}{1-n}}.y=0, \qquad \frac{d^2y}{dt^2}+\frac{2}{t}\cdot\frac{dy}{dt}+\mathrm{B}t^{\frac{2n-3}{1-n}}.y=0 \ldots. (14, 15).$$

Hence, remembering the Lemma (Solution of Quest. 1905), (15) shows that $f(1)=0$ is connected with $f\left(-2-\frac{1}{1-n}\right)=0$, while from (14) we see that $f(1)=0$ is connected with $f\left(-2+\frac{1}{1-n}\right)=0$; n being an integer either positive or negative, and the negative sign indicating integrations instead of differentiations. Let $[(a), (b)]$ denote the transformation of $f(a)$ into $f(b)$, which involves the possibility of transforming $f(b)$ into $f(a)$. Then we may add to the transformations indicated in the Solution of Quest. 1905 the following (n being an integer)

$$\left[(1), \left(-2+\frac{1}{1-n}\right)\right], \quad \left[(-1), \left(-2-\frac{1}{1-n}\right)\right] \quad \ldots (16, 17).$$

Let $n=0$, then (16) furnishes us with the transformation $[(1), (-1)]$, and (17) with $[(1), (-3)]$; let $n=2$, then (17) gives $[(1), (-1)]$, and (16) gives $[(1), (-3)]$. So, again, taking one of the results of the solution just referred to, viz. the general transformation which, in the notation of this paper, is expressed by $\left[(m), \left(-\frac{m}{m+1}\right)\right]$, we find that $m=1$ gives $\left[(1), \left(-\frac{1}{2}\right)\right]$, and $m = \frac{1}{2}$ gives $\left[\left(\frac{1}{2}\right), \left(-\frac{1}{3}\right)\right]$; so, applied to another result of that solution, viz. $\left[(m), \left(-\frac{3m+4}{m+1}\right)\right]$, $m = -\frac{3}{2}$ gives $\left[\left(-\frac{3}{2}\right)(-1)\right]$, while $m = -\frac{1}{2}$ gives $\left[\left(-\frac{1}{2}\right), (-5)\right]$. It might be interesting to inquire into the extent of these transformations. For instance, is it possible to transform $f(1) = 0$ into $f(2) = 0$?

[In the Solution of Quest. 1905, for $1-m$ in eq. (9) read $1+m$, also for at^k in eq. (13) read $at^k y$, and for $ax^{\frac{-m}{m+1}}$ in eq. (14) read $ax^{\frac{-m}{m+1}}y$.]

2291. (Proposed by Professor SYLVESTER.)—Prove that

$$\frac{[-\log(1+x)]^n}{\Pi n} = \frac{x^n}{\Pi n} + S_{n,1}\frac{x^{n+1}}{\Pi(n+1)} + S_{n+1,2}\frac{x^{n+2}}{\Pi(n+2)}$$

$$+ S_{n+2,3}\frac{x^{n+3}}{\Pi(n+3)} + \dots \quad \text{i.e.} = \sum_{\mu=\infty}^{\mu=0}\left(S_{n+\mu-1,\mu}\frac{x^{n+\mu}}{\Pi(n+\mu)}\right),$$

where $S_{i,j}$ means the arm of the j-ary products of the numbers $1, 2, 3, \dots i$.

Solution by GABRIEL TORELLI.

Soit u un nombre entier positif, et soit $S_{j,i}$ le coefficient de t^j dans le développement du produit $(1+t)(1+2t)(1+3t)\dots(1+it)$, on aura

$$\frac{[\log(1+x)]^m}{\Pi m} = \frac{x^m}{\Pi m} - S_{1,m}\frac{x^{m+1}}{\Pi(m+1)} + S_{2,m+1}\frac{x^{m+2}}{\Pi(m+2)} - \dots$$

Le théorème du binôme donne

$$(1+x)^\beta = \sum_r \binom{\beta}{r} x^r = \sum_r \beta(\beta-1)(\beta-2)\dots[\beta-(r-1)]\frac{x^r}{\Pi r},$$

mais on a $\beta(\beta-1)(\beta-2)\dots[\beta-(r-1)] = \sum_m (-1)^{r-m} S_{r-m,r-1}\beta^m$,

$$\text{donc } (1+x)^\beta = \sum_m \left(S_{0,m-1}\frac{x^m}{\Pi m} - S_{1,m}\frac{x^{m+1}}{\Pi(m+1)} + S_{2,m+1}\frac{x^{m+2}}{\Pi(m+2)} - \dots\right.$$

$$\left.\dots + (-1)^{r-m} S_{r-m,r-1}\frac{x^r}{\Pi r} + \dots\right)\beta^m,$$

et puisque on a d'ailleurs $(1+x)^\beta = \epsilon^{\beta\log(1+x)} = \sum_m \frac{[\log(1+x)]^m}{\Pi m}\beta^m$,

on en conclut $\frac{[\log(1+x)]^m}{\Pi m} = \frac{x^m}{\Pi m} - S_{1,m}\frac{x^{m+1}}{\Pi(m+1)} + S_{2,m+1}\frac{x^{m+2}}{\Pi(m+2)} - \dots$

$$\dots\dots + (-1)^{r-m} S_{r-m,\,r-1}\frac{x^r}{\Pi r} + \dots\dots\dots\dots (1).$$

A cette égalité se borne la Question 2291; mais comme l'illustre Prof. **Sylvester** a ensuite proposé sur le *Giornale ad uso degli studenti delle università Italiane* de déduire de l'équation (1) que $S_{j,\,i}$ est divisible pour tous les nombres premiers plus grands que $j+1$ et qui se trouvent dans la série $i+1,\ i,\ i-1,\ \dots,\ i-j+1$, je ne crois pas mal à propos d'adjoindre la démonstration de ce dernier théorème. Par un théorème dû à Eisenstein (Bertrand, *Calcul Différentiel*, No. 332), on connaît qu'étant donnée une série $u = A_0 + A_1x + A_2x^2 + \dots$, si l'on désigne avec $B_{m,\,n}$ le coefficient de x^n en u^m, on devra avoir $B_{m,n} = \sum_{s=1}^{s=n}(-1)^{n-s}\binom{m}{s}\binom{m-s-1}{n-s}A_0^{m-s}B_{s,n}$ (2);

par conséquent si l'on pose $u = \frac{\log(1+x)}{x} = 1 - \frac{1}{2}x + \frac{1}{3}x^2 + \dots + \frac{(-1)^n}{n+1}x^n + \dots$ on aura à cause de l'équation (1) $B_{m,n} = (-1)^n\frac{\Pi m}{\Pi(n+m)}S_{n,\,n+m-1}$, et l'équation (2) deviendra

$$(-1)^n\frac{\Pi m}{\Pi(n+m)}S_{n,\,n+m-1} =$$

$$\sum_{s=1}^{s=n}(-1)^{n-s}\binom{m}{s}\binom{m-s-1}{n-s}(-1)^n\frac{\Pi s}{\Pi(n+s)}S_{n,\,n+s-1},$$

laquelle, en observant que l'on a

$$\frac{\Pi s\,.\,\Pi(n+m)}{\Pi m\,.\,\Pi(n+s)}\binom{m}{s} = \binom{m-1}{s-1}\binom{n+m}{n+1} : \binom{n+s}{n+1},$$

pourra être écrite sous la forme

$$\frac{S_{n,\,n+m-1}}{\binom{n+m}{n+1}} = \sum_{s=1}^{s=n}\binom{m-1}{s-1}\binom{m-s-1}{n-s}\frac{S_{n,\,n+s-1}}{\binom{n+s}{n+1}} \dots\dots\dots\dots (3),$$

et celle-ci, si $m < n$, se réduit à

$$\frac{S_{n,\,n+m-1}}{\binom{n+m}{n+1}} = \sum_{s=1}^{s=m-1}\binom{m-1}{s-1}\binom{m-s-1}{n-s}\frac{S_{n,\,n+s-1}}{\binom{n+s}{n+1}} \dots\dots\dots (4),$$

puisque dans l'équation (3) le symbole $\binom{\mu}{\nu}$, comme pour l'ordinaire, doit être considéré comme égal à zero si μ ou ν sont négatifs, et égal à t si μ ou ν sont nuls.

Or l'équation (4) nous fait voir que, si de $s = 1$ jusqu'à $s = m-1$ $S_{n,\,n+s-1} : \binom{n+s}{n+1}$ est entier, tel sera aussi $S_{n,\,n+m-1} : \binom{n+m}{n+1}$; et puisque

$S_{n,n} : \binom{n+1}{n+1}$ est entier, on peut conclure que pour chaque valeur de m plus petite que n, $S_{n,n+m-1} : \binom{n+m}{n+1}$ est entier; mais alors chaque terme du second membre de l'équation (3) étant entier, tel sera aussi le premier; d'où il résulte que, pour toute valeur de m, $S_{n,n+m-1} : \binom{n+m}{n+1}$ est entier; c'est-à-dire, en posant $n=j$, $n+m-1=i$, on pourra énoncer que $S_{j,i}$ est divisible pour tous les nombres plus grands que $j+1$, et qui se trouvent dans la série $i+1, i, i-1, \ldots\, i-j+1$; c'est en ça que consiste le théorème de SYLVESTER.

2332. (Proposed by Professor SYLVESTER.)—Show that but two Cartesians can be drawn having a given axis and passing through any four given points, and prove thereby, with the aid of the proposition in Quest. 1990, that all the Cartesians which can be drawn through four given concyclic points consist exclusively of two tribes (*families of families*), whose foci lie respectively on the two circular cubics, of which the four given points are foci.

I. *Solution by the* PROPOSER.

1. The function of a Cartesian whose axis is the axis of x is well known to be a parabolic quadratic form in (x^2+y^2, x), that is, putting r for x^2+y^2, its equation is $A^2r^2+2ABxr+B^2x^2+Cr+Dx+E=0$. Hence if four points are given, C, D, E may be eliminated linearly, and a quadratic equation will give the value of the ratio A : B, each of which will correspond to one system of values of C, D, E; which proves the first part of the theorem.

2. If possible, let F, G, H, three points not all on the same one circular cubic having the given points A, B, C, D for concyclic foci, be foci of a Cartesian through A, B, C, D, and produce F, G, H to meet the two cubics, of which A, B, C, D are the foci, in F′, G′, H′; F_1, G_1, H_1; then, no matter whether the intersections be all real or not, two Cartesians can be drawn, having the above two systems of points for their foci respectively [see Solutions of Quest. 1990, *Reprint*, Vol. VI., p. 35, &c.]; thus there will be three coaxial Cartesians passing through four given points, which is contrary to (1). Hence every focus of any Cartesian through the four concyclic points must lie on one of the corresponding circular cubics, and moreover all three foci in the same cubic, as was to be proved.

3. We are thus led to notice a paradox even more strange than that so ably explained by Professor CAYLEY in his masterly solution of Quest. 1990. (For the proposer's method of solution see *Philosophical Mag.* for 1866.)

The paradox consists in the mere fact, that there is a definite linear locus to the foci of systems of curves containing two arbitrary parameters, which we should at first sight suppose, must in all cases either be capable of occupying any position, real or imaginary, in a plane, or at all events (which seems the more difficult supposition to admit) of ranging over the plane within certain real and imaginary limits. It is worthy of inquiry, whether as regards Cartesians through four non-concyclic points, the position of the foci is or not absolutely unlimited. Another mode of stating the proposition, is to say that, whatever Cartesian is drawn through four concyclic points, each focus respectively coincides with the focus of some one conic that can be drawn through the same four points. The paradox here referred to, and equally the one insisted upon by Prof. CAYLEY, both appear to be of the nature of *Anti-porisms;* conceiving a porism to signify a case of a determinate problem becoming indeterminate, or an impossible one possible, or a more determinate one less so; here we have the reverse phenomenon of constructions usually possible absolutely failing, or from indeterminate becoming determinate; a simple instance of the same kind is noticed in my paper on this very subject in the *Phil. Mag.*, viz., the case of the position of the axis of conics passing through four points which, from being arbitrary in general, becomes fixed to two positions when the four points lie in a circle.

II. *Solution by* M. W. CROFTON, B.A.

The theorem may be otherwise enunciated as follows:—

The locus of the foci of all Cartesian ovals which can be drawn through four given concyclic points is identical with the locus of the foci of the conics which pass through the same four points; viz., the two circular cubics of which those points are foci.

This extremely remarkable theorem states an apparent absurdity. As six conditions are required to determine a Cartesian oval, if we are given four points on the curve, we might apparently take any point whatever for a focus; and in fact this is true in general, provided the four points do not lie on a circle. To ask for the locus of the foci of the Cartesians passing through four given points appears as absurd as to ask for that of the foci of the *conics* passing through three points. But when the four points lie on a circle, although two conditions are still left free, so that the assemblage of Cartesians are not strictly a *family,* but are more correctly termed a *family of families,* the foci are partially determined, being restricted to the circular cubics above mentioned.

The following proof was suggested to me by the use made of Möbius' and Feuerbach's theorems by Mr. BURNSIDE, in his Solution of Quest. 1990. [*Reprint,* Vol. VI., p. 100.]

Call the four points 1, 2, 3, 4. Let R, S be two foci of any Cartesian passing through them, whose equation is

$$\rho = k\sigma + l.$$

Let $\rho_1, \rho_2, \rho_3, \rho_4$ be the distances of R from 1, 2, 3, 4; and $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ those of S; then since 1, 2, 3, 4 are concyclic, we have

$$(234)\,\rho_1^2-(134)\,\rho_2^2+(124)\,\rho_3^2-(123)\,\rho_4^2=0;$$

putting for ρ_1 its value $k\sigma_1+l$, for ρ_2, $k\sigma_2+l$, &c., this becomes

$$\left.\begin{array}{l} k^2\left\{(234)\,\sigma_1^2-(134)\,\sigma_2^2+(124)\,\sigma_3^2-(123)\,\sigma_4^2\right\} \\ +\,2kl\left\{(234)\,\sigma_1-(134)\,\sigma_2+(124)\,\sigma_3-(123)\,\sigma_4\right\} \\ +\,l^2\left\{(234)-(134)+(124)-(123)\right\} \end{array}\right\}=0.$$

Of these three terms, the third is identically 0, and the first vanishes by Feuerbach's theorem; hence

$$(234)\,\sigma_1-(134)\,\sigma_2+(124)\,\sigma_3-(123)\,\sigma_4=0,$$

which is Möbius' relation between the distances of 1234 from a focus of a conic passing through them. Hence Möbius' theorem is true for the distances of four points from *any focus of a Cartesian* passing through them, provided the four points are concyclic.

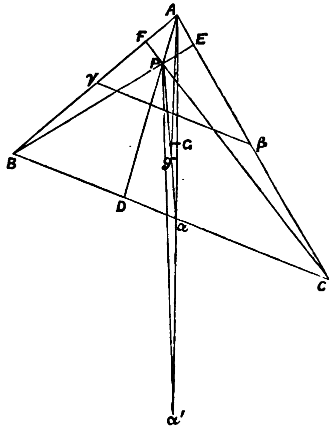
The question is thus reduced to Quest. 1990; (see Mr. BURNSIDE's Solution).

An interesting case of the theorem is when the four given points lie on a straight line. Here it seems to fail for the conics, as the family of conics reduces to the straight line itself. However, as every point in the plane is a focus to the straight line, considered as a hyperbola, of course every point on the circular cubic is one. But an infinite number of Cartesians (of the *indented* form) will pass through four points on a straight line; and the locus of their foci will be the two circular cubics (of a particular kind) whose foci are those points. I am not aware of the variety of circular cubic whose foci are collinear having yet been considered.

I have given some further researches on the subject, including an extension of the above method to bi-circular quartics through 4 concyclic points, in a paper just published in the *Proceedings* of the London Mathematical Society.

2328. (Proposed by J. GRIFFITHS, M.A.)—(1.) Let G be the centre of gravity of any triangle ABC; β and γ the middle points of two of its sides (AB, AC, say); show that the circles described upon the lines AG, $\beta\gamma$, BC as diameters, are coaxal. Hence, (2.) if P be the point of intersection of the perpendiculars of the triangle: α the middle point of the side BC; g a point on αP such that $\alpha g = \frac{1}{3}\alpha$P; also α' a point on Aα produced, such that A$\alpha' =$ 2Aα; then the nine-point circle of the triangle is coaxal with the circles described upon Ag and Pα' as diameters.

Solution by H. TOMLINSON; J. DALE; W. CHADWICK; *and others.*



1. If we take α as origin of rectangular coordinates, and Aα as axis of x, it may easily be proved that, if we put BC $= a$, and A$\alpha = l$, the equation of the circles on BC, $\beta\gamma$, Aα as diameters are respectively

$$x^2+y^2=\tfrac{1}{4}a^2 \;\ldots\ldots\ldots\; (1),$$
$$(x-\tfrac{1}{2}l)^2+y^2=\tfrac{1}{16}a^2 \;\ldots\ldots\; (2),$$
$$(x-\tfrac{2}{3}l)^2+y^2=\tfrac{1}{9}l^2 \;\ldots\ldots\ldots\; (3);$$

therefore the radical axis of (1) and (2) is the same as the radical axis of (2) and (3), and consequently (1), (2), and (3) are coaxal.

2. Next, in the triangle APα': since α is the middle point of Aα', and g the centre of gravity of the triangle, the circles described on Ag and Pα' as diameters will be coaxal with the circle described on the line joining α and the middle point of AP, which circle passes through D (since the angles at D are right angles), and is therefore the nine-point circle of the triangle ABC.

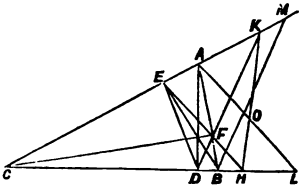
EXTRACT FROM TABLE OF FACTORS OF 100000 MILLIONS.

By W. BARRETT DAVIS.

NUMBERS.	PRIME FACTORS.
100000000957	487 . 7459 . 27529
61	7 . 401 . 35625223
63	53 . 79 . 23883449
67	23 . 167 . 26034887
69	11 . 11 . 826446289
73	31 . 37 . 313 . 278543
79	577 . 173310227
81	10687 . 9357163
87	83 . 1999 . 602711
91	11 . 9090909181
93	19 . 19 . 113 . 2451401
97	13 . 769 . 10003001
99	4999 . 20004001
100000001003	7 . 7 . 1429 . 1428143
1009	Not divisible by any number < 44001.
11	587 . 170357753
17	7 . 17 . 43 . 631 . 30971
21	6661 . 15012761
23	13 . 7692307771
27	347 . 829 . 347629
29	647 . 7013 . 22039
33	10303 . 9705911
39	4861 . 20571899
41	127 . 787401583
47	37 . 2702702731
51	17 . 2269 . 2592487
53	227 . 419 . 677 . 1553
57	11 . 163 . 55772449
59	7 . 23 . 47 . 13215277
63	17957 . 5568859
69	19 . 53 . 99304867
71	29 . 643 . 5362793
77	9803 . 10200959
81	2939 . 34025179
83	263 . 380228141
87	7 . 25847 . 552703
89	367 . 272479567
100000001093	523 . 191204591

2284. (Proposed by J. Griffiths, M.A.)—(1.) Q is the point of intersection of the polars of any point P with respect to two given similar and similarly situated conics; show that the locus of the middle point of the line PQ is the radical axis of the curves in question. Hence, (2) if D, E, F be the feet of the perpendiculars of any triangle ABC; and AL, BM, CN be lines drawn through A, B, C parallel to EF, FD, DE, and cutting BC, CA, AB in the points L, M, N; it is required to prove that the axis of homology of the triangles ABC, DEF passes through the middle points of the segments AL, BM, CN.

Solution by W. Chadwick; J. Dale; H. Tomlinson; *and others.*



1. The two conics may be regarded as circumscribing a quadrilateral, of which two opposite sides are the radical axis and the line at infinity, P being the intersection of the other two sides, and Q the intersection of the diagonals; consequently PQ is divided harmonically by the radical axis and the line at infinity, that is, PQ is bisected by the radical axis.

Otherwise: Let the equations of the conics be

$$ax^2+hxy+by^2+2gx+2fy+c=0,\quad ax^2+2hxy+by^2+2g'x+2f'y+c'=0;$$

then the equation of their radical axis is

$$2x(g-g')+2y(f-f')+c-c'=0 \;\ldots\ldots\ldots\ldots\ldots\ldots(1);$$

also, since the equations of the two polars of any point (x', y') are

$$axx'+h(x'y+y'x)+byy'+g(x+x')+f(y+y')+c=0,$$

$$axx'+h(x'y+y'x)+byy'+g'(x+x')+f'(y+y')+c'=0;$$

therefore $$(x+x')(g-g')+(y+y')(f-f')+c-c'=0\;\ldots\ldots\ldots(2);$$

and since the coordinates of the middle points of the lines PQ are $\frac{1}{2}(x+x')$, $\frac{1}{2}(y+y')$; therefore, from (2), they lie on (1).

2. The equations of EF, FD, DE are respectively

$$-\cos A\,.\,x+\cos B\,.\,y+\cos C\,.\,z=0,\quad \cos A\,.\,x-\cos B\,.\,y+\cos C\,.\,z=0,$$
$$\cos A\,.\,x+\cos B\,.\,y-\cos C\,.\,z=0;$$

therefore the equation of the axis of homology of the triangles ABC, DEF is $\cos A\,.\,x+\cos B\,.\,y+\cos C\,.\,z=0$, which is the equation of the radical axis of the circles ABC, DEF.

The equations of AL, BM, CN are

$$cy+bz=0,\quad az+cx=0,\quad bx+ay=0;$$

therefore AL, BM, CN are the several polars of A, B, C with respect to the circle ABC.

The polars of A, B, C with respect to the circle DEF are

$$-2a\cos A\,.\,x+cy+bz=0,\quad cx-2b\cos B\,.\,y+az=0,\quad bx+ay-2c\cos C\,.\,z=0,$$

and these lines pass respectively through L, M, N; therefore L, M, N are the intersections of the polars of A, B, C with respect to the circles ABC, DEF, which are similar and similarly situated conics; hence AL, BM, CN are bisected by the radical axis $\cos A \, . \, x + \cos B \, . \, y + \cos C \, . \, z = 0$.

Otherwise: Let EF, DF meet CB, CA in H, K, respectively; and let O be the middle point of AL; then, in the triangle ACL, we have

$$\frac{AO}{LO} \cdot \frac{LH}{CH} \cdot \frac{CK}{AK} = +\frac{1}{1} \cdot \frac{AE}{CE} \cdot \frac{CE}{AE} = +1;$$

therefore HK, the axis of homology of the triangles ABC, DEF passes through O, the middle point of AL; and similarly of BM and CN.

2295. (Proposed by G. C. De Morgan, M.A.) — To express the values of $\frac{1}{4}\pi$ and $\log_e (2)$ in the form of continued fractions.

Solution by J. Dale; E. McCormick; *and others.*

Let $$F(m) = \frac{1}{m} - \frac{1}{m+n} + \frac{1}{m+2n} - \frac{1}{m+3n} \ldots\ldots,$$

and $$F(m+n) = \; + \frac{1}{m+n} - \frac{1}{m+2n} + \frac{1}{m+3n} \ldots\ldots,$$

then $$F(m) + F(m+n) = \frac{1}{m} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Assume $F(m) = \dfrac{1}{m + f(m)}$, and $F(m+n) = \dfrac{1}{m+n+f(m+n)}$;

then, dividing (1) first by $F(m)$ and next by $F(m+n)$, we have

$$\frac{F(m+n)}{F(m)} = \frac{f(m)}{m}, \quad \text{and} \quad \frac{F(m)}{F(m+n)} = \frac{n + f(m+n)}{m} \quad \ldots\ldots\ldots (2, 3).$$

From (2) and (3) we have

$$f(m) = \frac{m^2}{n + f(m+n)}, \; f(m+n) = \frac{(m+n)^2}{n + f(m+2n)}, \; f(m+2n) = \frac{(m+2n)^2}{n + f(m+3n)}, \; \&c.$$

$$\therefore \quad F(m) = \frac{1}{m + f(m)} = \cfrac{1}{m + \cfrac{m^2}{n + f(m+n)}} = \cfrac{1}{m + \cfrac{m^2}{n + \cfrac{(m+n)^2}{n + f(m+2n)}}}$$

$$\ldots\ldots\ldots$$

$$= \frac{1}{m+} \, \frac{m^2}{n+} \, \frac{(m+n)^2}{n + f(m+2n) +} \, \frac{(m+2n)^2}{n + f(m+3n) + \&c.}$$

Suppose $m = 1$, $n = 2$; then $F(m) = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}$,

therefore $$\frac{\pi}{4} = \frac{1}{1+} \, \frac{1^2}{2+} \, \frac{3^2}{2+} \, \frac{5^2}{2+} \, \frac{7^2}{2 + \&c.}.$$

Suppose $m=1$, $n=1$; then $F(m) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \log (2)$;

therefore $$\log_e (2) = \frac{1}{1+}\,\frac{1^2}{1+}\,\frac{2^2}{1+}\,\frac{3^2}{1+}\,\frac{4^2}{1+\&c.}.$$

[Another method of investigating these continued fractions is given in Art. 421 of Bertrand's *Traité de Calcul Différentiel*, where the fraction for $\frac{1}{4}\pi$ is called "formule célèbre donnée pour la première fois sans démonstration par Brounker." See also Hymers' *Trigonometry*, p. 157 (ed. 1847.)]

1995. (Proposed by J. GRIFFITHS, M.A.)—Let G be the centre of gravity of any triangle ABC; I the centre of the inscribed circle; $\lambda_1, \lambda_2, \lambda_3$ the points where the sides BC, CA, AB are touched by the escribed circles opposite A, B, C, whose centres are I_1, I_2, I_3; Q the point of intersection of $A\lambda_1$, $B\lambda_2$, $C\lambda_3$; Q′ the point inverse to Q; G′ the inverse of G; then the polars of I, G′, Q′, with respect to the triangle, meet in the same point. Also, if these respective polars be denoted by P_1, P_2, P_3, and the harmonic conjugate of P_3, with respect to P_1, P_2, by P_4; show that the points of intersection of the circles (ABC), $(I_1I_2I_3)$ with P_3, P_4 respectively, lie on another circle which touches the circle (I).

Solution by JAMES DALE.

Taking ABC as triangle of reference, and putting s_1 for $s-a$, s_2 for $s-b$, and s_3 for $s-c$, the coordinates of the various points and equations of the lines and circles will be as follows:—

$$(G, I) \ldots\ldots\ldots\; ax = by = cz = \frac{2\Delta}{3}, \quad x = y = z = \frac{2\Delta}{s};$$

$$(I_1, I_2, I_3) \ldots\; -x = y = z = \frac{2\Delta}{s_1},\; x = -y = z = \frac{2\Delta}{s_2},\; x = y = -z = \frac{2\Delta}{s_3};$$

$$(Q, Q') \ldots\ldots\; \frac{ax}{s_1} = \frac{by}{s_2} = \frac{cz}{s_3} = \frac{2\Delta}{s},\; \frac{s_1}{a}x = \frac{s_2}{b}y = \frac{s_3}{c}z = \frac{2\Delta}{a^2s_1^{-1} + b^2s_2^{-2} + c^2s_3^{-3}}$$

$$(G') \ldots\ldots\; \frac{x}{a} = \frac{y}{b} = \frac{z}{c} = \frac{2\Delta}{a^2+b^2+c^2};$$

$$(P_1, P_2, P_3) \ldots\; x+y+z = 0,\; \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 0,\; \frac{s_1}{a}x + \frac{s_2}{b}y + \frac{s_3}{c}z = 0.$$

Since $(P_3) \equiv s(P_2) - (P_1)$, the lines P_1, P_2, P_3 meet in the same point, and the harmonic conjugate of P_3 with respect to P_1, P_2 must be of the form $s(P_2) + P_1$, hence the equation to P_4 is

$$(P_4) \ldots\ldots\ldots\ldots\ldots\ldots\; \frac{s+a}{a}x + \frac{s+b}{b}y + \frac{s+c}{c}z = 0;$$

also circle (ABC) .... $ayz + bzx + cxy = 0$,

and circle $(I_1I_2I_3)$ .... $ax^2 + by^2 + cz^2 + (a+b+c)(yz+zx+xy) = 0$,

or circle $(I_1I_2I_3)$ .... $ayz + bzx + cxy + (ax+by+cz)(x+y+z) = 0$.

This shows that the line $x+y+z=0$ is the radical axis of the circles (ABC), $(I_1I_2I_3)$; and since (P_3), (P_4) meet in a point on the radical axis, the points of intersection of (P_3) with (ABC) and (P_4) with $(I_1I_2I_3)$ lie on a circle. The equation of this circle is

$$ayz+bzx+cxy-\tfrac{1}{2}(ax+by+cz)\left(\frac{s_1}{a}x+\frac{s_2}{b}y+\frac{s_3}{c}z\right)=0,$$

which may also be written

$$xyz+bzx+cxy+(ax+by+cz)(x+y+z)$$
$$-\tfrac{1}{2}(ax+by+cz)\left(\frac{s+a}{a}x+\frac{s+b}{b}y+\frac{s+c}{c}z\right)=0.$$

This circle touches the inscribed circle (I), the equation of which may be written $ayz+bzx+cxy-(ax+by+cz)\left(\frac{s_1^2}{bc}x+\frac{s_2^2}{ca}y+\frac{s_3^2}{ab}z\right)=0,$

and the equation to the common tangent is

$$s_1(a-b)(a-c)x+s_2(b-c)(b-a)y+s_3(c-a)(c-b)z=0.$$

SOME ADDITIONAL OBSERVATIONS ON THE FOUR-POINT PROBLEM.

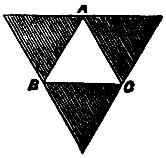
BY W. S. B. WOOLHOUSE, F.R.A.S.

The geometrical exercise in probabilities, familiarly known as the "Four-point problem," was first proposed for solution in the *Educational Times* for April, 1864, as follows:—

"Question 1491. (Proposed by Professor SYLVESTER.)—Show that the chance of four points forming the apices of a reentrant quadrilateral is $\frac{1}{4}$ if they be taken at random in an indefinite plane, but $\frac{1}{4}+e^2+x^2$, where e is a finite constant and x a variable quantity, if they be limited by an area of any magnitude and of any form."

In the number for September, 1865, (Note to Question 1805) it is explained that the limiting area is assumed to be of a convex form.

The assigned value $\frac{1}{4}$ is obtained by a remarkably simple and apparently efficacious method of solution propounded by Professor Sylvester, who is not, however, an advocate of the principle involved in it. This method, for which I am indebted to Professor Cayley, may be briefly stated thus. [See Prof. Sylvester's Memoir "On the real and imaginary roots of equations," in the *Philosophical Transactions* for 1864, p. 653, note *b*.]



Suppose A, B, C are the three points which form the largest triangle. Then, completing the diagram by parallels as shown in the margin, the fourth point D is by hypothesis in one of the four equal triangles; and the figure is evidently convex or concave according as D is in one of the shaded triangles or in the original triangle ABC, *i.e.* the chance of a concave figure is 1 in 4.

Professor Cayley and Professor Sylvester are satisfied of the insufficiency of this solution;* and the latter objects that it is possible, by an analogous and equally good argument, to obtain an inconsistent result. Suppose that A is the point which is most distant from a certain fixed point O: then the four points lie within the circle, radius OA, or the chance in general should be the same as if the four points were taken, one of them on the circumference of, and the others within, a given circle: which gives a chance different from $\frac{1}{4}$.

It will be found that the latter of these methods is quite legitimate when the four points are limited to the area of a circle of any proposed magnitude.

For the accurate investigation of all questions of this kind involving direct values of probability, two conditions are absolutely essential to the usual process of calculation.

1. All the possible ways, both for and against the specified event, should individually possess an equal degree of probability.

2. In estimating relatively the number of ways which fulfil the proposed event, and the total number of ways, it is indispensable that all the equally probable elements shall be alike included, without exception, or severally repeated an equal number of times.

When these two conditions subsist, it is evident that the required value of the sought probability is correctly expressed by the fraction

$$\frac{\text{Number of ways fulfilling the event}}{\text{Total number of ways}}.$$

If, however, either of the two conditions be wanting, it is equally evident that the reasoning entirely fails, and therefore that the truth of the result cannot in such case be relied upon. It will hence be sufficient to remark that the second condition especially is not to be found in the first mentioned solution by Professor Cayley, which affords no evidence whatever of involving all possible combinations alike. This solution would, indeed, be thoroughly conclusive if it could be shown that the grand total of cases admitted of being analysed into distinct systems or groups, analogous to that which is embodied in the diagram when the fourth point Q ranges over the entire figure. But it would not be difficult to show that this is impossible.

A similar remark is applicable to the solution by Mr. G. C. De Morgan, (*Reprint*, Vol. V., p. 109,) for all points on an indefinite plane cannot be made up of collections of points lying on arcs of hyperbolas, nor yet of collections lying on an indefinite number of straight lines connecting every pair of points.

Mr. Wilson's neat solution, (*Reprint*, Vol. V., p. 81,) without being in strict accordance, conforms so nearly with the requisite conditions of the investigation, that the numerical result arrived at must necessarily be near the truth. The corresponding solution to his interesting four-line problem, it is needless to say, is strictly legitimate. That of the four-point problem would be so, if each point were determined by the intersection of an independent pair of lines. By assuming the points to be determined from the intersections of four, in lieu of eight, random lines, the calculation is immediately accomplished, but a sort of correlation is thus established amongst the points, which unavoidably interferes with the required arbitrary character of their positions. That this is the case, might be inferred from the three-point problem. If the three points are here supposed to be the intersections of three random lines, the probability of an acute triangle has been conclusively shown to be $\frac{1}{4}$, (*Reprint*, Vol. V., p. 70,) whereas the probability is known to have a different value when the three points are quite independent. In my former observations (*Reprint*, Vol. VI., p. 52) I have indicated an accurate solution of the four-point problem for any area, and shown as an example that when the points are limited to the surface of any triangle Mr. Wilson's result ($\frac{1}{3}$) is then exact. When the four points are taken on the area of a rectangle, the probability is then $\frac{11}{36}$, and not $\frac{5}{18}$ as stated by Professor Sylvester in Quest. 1846. (*Educational Times* for November, 1865.)

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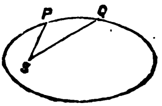
* See the Note to Question 1832, *Educational Times* for October, 1865, p. 166.

The method of solution sketched out by Dr. Ingleby (*Reprint*, Vol. V., p. 82) is analogous to that given by Professor De Morgan, and is equally affected by the imperfection pointed out in my former observations. In both of these solutions it may also be objected that in comparing the infinite areas one of the points is tacitly assumed to be posited centrally on the indefinite plane; a supposition manifestly inconsistent with the premises, that it should in common with the other three points occupy every position whatever on that plane.

I have now only to briefly advert to what Mr. Godfray has advanced (*Reprint*, Vol. VII., p. 65) in support of his former statement (*Reprint*, Vol. VI., p. 72) about what he supposes to be a want of definiteness in the word "random," as it occurs in the enunciation of geometrical chance questions. So far as my experience extends, I have observed amongst mathematicians a common consent as to the proper and precise meaning of the term, and have only been able to arrive at the conclusion that discordant results in solving questions on chance are in general not attributable to any such cause as that which Mr. Godfray alleges. I also agree with Mr. Wilson that Mr. Godfray's idea of a "random line drawn according to a special law" is deficient in intelligibility and consistency, and would certainly be more correctly stated in other terms, conveying the mechanical conditions under which the line is supposed to be drawn.

As an example, Mr. Godfray gives the following problem. A line is drawn at random from the focus of an ellipse to the perimeter, what is the chance that its length is greater than the semi-major axis?



Let SP, SQ be any two focal lines. Then, according to Mr. Godfray, a random radius may be defined to be

Firstly: A line whose chance of falling within the angle PSQ is proportional to the angle PSQ.

Secondly: A line whose chance of falling on the arc PQ is proportional to the length of the arc PQ.

Thirdly: A line whose chance of falling on the area PSQ is proportional to that area.

Now, if we take into consideration the lines so drawn from the focus, the first case truly represents "a line drawn at random;" the second is a line drawn to a random point in the periphery; the third is a line drawn through a random point within the area of the ellipse; and whichever of these operations is meant, the same ought to be so expressed. According to the original enunciation of the problem, instead of sharing in Mr. Godfray's puzzle as to which of the three cases is the most natural, I should have no hesitation whatever in pronouncing upon that which is first mentioned, and I apprehend that so obvious a decision would undoubtedly be arrived at by mathematicians in general.

Respecting a "random point," I have only to add, that the accurate definition, stated by Mr. Wilson, that it is one equally likely to fall in any one of the small equal areas into which space is supposed to be divided, simply negatives the imposition of any restriction, and only expresses in another form the universally recognised condition, that all points on the surface are equally available, and that the accident of position is not fettered by any law whatever.

NOTE ON LOCAL PROBABILITY. BY M. W. CROFTON, B.A.

Some remarks which possibly may help to clear up certain obscurities in this subject may not be unacceptable. I would premise that the points I am touching on need much ampler developments than the limits of a Note permit; so that much must be left to the sagacity of the reader.

The expression "*at random*" has in common language a very clear and definite meaning; one which cannot be better conveyed than by Mr. Wilson's definition,—"*according to no law*;" and in this sense alone I mean to use it in this Note. It has thus very wide applications, many of them altogether beyond the province of mathematical measurement or calculation.

In Mathematical Probability, which consists essentially in arithmetical calculation, when we speak of a thing of any kind taken at random, there is always a direct reference to the *assemblage of things* to which it belongs and from which it is taken at random, or according to no law; which here comes to the same thing as saying, that any one is as likely to be taken as any other. Before we can attempt to sum up the favourable and unfavourable cases in any problem, we must first have a clear conception of what the assemblage is from which we take.

When, therefore, in questions on Local Probability, we speak of a point or line taken at random, *i.e.* according to no law, there is no obscurity in the idea; but it cannot be made a matter of arithmetical calculation till we have a definite conception of the nature of the assemblage of points or lines one (or more) of which we take.

A point, then, is taken at random on a plane. What is the assemblage of points we select it from? Certainly not (as might hastily be answered) "all the points in the plane." To me it is impossible to form any conception of all the mathematical points in a plane considered as a sum. Take points by millions, we are as far from filling up the smallest space on the plane as when we began. That which has no magnitude cannot fill up space.

To obtain the true view of the nature of the assemblage we are dealing with, I take a simple example. The chance of a point taken at random on a chess-board falling on a white square is $\frac{1}{2}$. What is the precise meaning of this? It means that if we make a number of trials, *i.e.*, take *a number of points at random*, and continually increase the number, till it becomes infinite,—ultimately the points on the white will be half the entire number. Hence we see at once that the true description of the assemblage from which we take is, *an infinity of points all taken at random in the plane.*

Let us examine the nature of this assemblage. As the points continue to be scattered at random over the plane, their density tends to become uniform. It is evident, in fact, that a random point is as likely to be in any element dS of the surface as in any equal element dS':* and therefore, by continuing to multiply points, the number in dS will be equal (or *subequal*, to use a term of Professor DE MORGAN'S) to that in dS'. Of course, though the density tends to become uniform, the disposition of the points does not tend to become symmetrical; those within any element dS will be scattered in the most irregular manner over that element.* However, it is important to remark that, *for all purposes of calculation*, the ultimate disposition may be supposed symmetrical: for as the position of any point is determined by that of the element dS, within which it falls, it matters not what arbitrary arrangement, within the element, we assume for the points. Hence we may assume that, if a point be taken at random in a plane, those from which it is taken are an infinite number symmetrically disposed over the plane.

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* Or, "the chance of the point falling on any area is proportional to the area." I cannot view this as an arbitrary definition (see Mr. Godfray's paper, *Reprint*, Vol. VII., p. 65) of a random point: we may call it a definition if we please, but it is a true definition, and any one incompatible with it is false. I look on it as a direct and self-evident deduction from the general definition that a random point follows no law. If the points *tended* to become denser in any part of the plane than in another, there must be some law attracting them there. We are reminded of the experiment of shaking steel filings from a height over a sheet of paper with a magnet beneath it.

Likewise, points taken at random in a line may be supposed equidistant. And if random values be taken for any *quantity*, they may be supposed to form an arithmetical series, with an infinitesimal difference.

Let us now consider the case of a straight line drawn at random in an infinite plane: the assemblage from which we select it, is as before, *an infinity of lines drawn at random in the plane.* What is the nature of this aggregate? First, since any direction is as likely as any other, as many of the lines are parallel to any given direction as to any other. Consider one of these systems of parallels; let them be cut by any infinite perpendicular. As this infinite system of parallels are drawn according to no law, they are as thickly disposed along any part of the perpendicular as along any other, *i.e.*, the intersections are random points on the perpendicular. Hence, referring to the last paragraph, it will be easy to see that, for all purposes of calculation, the assemblage of lines may be thus conceived:—Divide the angular space round any point into a number of equal angles $d\theta$, and for every direction let the plane be ruled with an infinity of parallel equidistant lines, the common infinitesimal distance being the same for every set of parallels. Or, draw from any centre an infinite system of equidistant concentric circles, and draw all the tangents at the points where they are met by a system of radii, diverging uniformly from the centre.

If we take any fixed axes in the plane, a random line may be represented by the equation $x \cos \alpha + y \sin \alpha = p$, where p, α are constants taken at random.

We may, if we please, take this as an analytical definition of a random line: and this appears to me to illustrate the true place of definitions in such a subject as the present. It has to be proved that the definition is a necessary consequence of the idea it represents. Were we to define a random line to be $y = mx + c$, where m, c are taken at random, it would be a false definition. Every line that can be drawn in the plane will of course be included in the equation, but all will not be equally likely to be taken; *e.g.*, it is easy to see that the axis of y is *infinitely* more likely to be taken than the axis of x.

In the problem treated by Mr. GODFRAY (*Reprint*, Vol. VI., p. 72) of two chords drawn at random in a circle, the want of definiteness, in my opinion, lies in the different senses which may be given to the expression, "drawing a chord." If it means "a line is drawn (at random) cutting the circle,"† it will give (as I think I have shown above) the second case Mr. Godfray mentions, all distances from the centre being equally probable. If it means "a (random) line drawn from a (random) point on the circle," or "a line joining two (random) points on the circle," it will give the first case. The two latter suppositions are identical here, as Mr. WOOLHOUSE has remarked.

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* Order thus results from disorder: the uniform density of the aggregate being unaffected by the disorder and irregularity of the arrangement of its ultimate constituents: much as a nebula of uniform brightness is related to the stars which compose it.

† There is no contradiction in subjecting a random entity to law, provided in some respect it is left free. Thus a random line may be subjected to a law, viz., it must meet the circle, but otherwise it follows no law—so that the above expression will mean, "A line taken at random from the random lines which meet the circle."

A random great circle on a sphere means a random plane through the centre; hence it will be easily seen that its pole is a random point on the sphere.

The points we are here discussing seem to be the first principles of a new and extensive subject, which offers great difficulties, and calls for much thought and study.

On the 3 and 4 point problems, I would observe that, whether they do or do not admit of solution in the case where the space is unlimited, one thing is clear, viz., that we must exclude all cases where any of the points are at a finite distance asunder, and reason only on the infinite cases—the former being infinitesimal in number compared to the latter. In fact, if two points are taken at random in an infinite plane or space, they *must be* at an infinite distance from each other—the chance of the contrary being infinitesimal.

On the Nine-point Circle. By W. Godward.

In any triangle ABC (Fig. 1), the feet of the perpendiculars AA_1, BB_1, CC_1 (which intersect in P), the middle points A_2, B_2, C_2 of the sides BC, CA, AB, and the middle points A_3, B_3, C_3 of the segments AP, BP, CP, are nine points which lie on the circumference of a circle, called the nine-point circle. Also if O be the centre of the circle circumscribing the triangle ABC, then the centre of the nine-point circle is the middle point of OP, and its radius is half the radius of the circle (O).

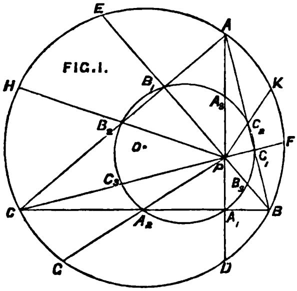


FIG. I.

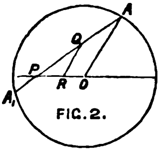


FIG. 2.

This problem of the nine-point circle is only a particular case of this simple lemma. If P be any point (Fig. 2) either within or without a circle whose centre is O, and A_1PA any chord, then if either segment of this chord, as AP, be bisected in Q, the locus of Q is a circle whose centre is the middle point of OP, and its radius half the radius of the original circle.

For, join OA and draw QR parallel to it; then since $PQ = QA$, we have by similar triangles $PR = RO$, and $RQ = \frac{1}{2}OA$, implying that the locus of Q is a circle whose centre is R and radius $= \frac{1}{2}OA$.

In Fig. 1, produce AA_1, BB_1, CC_1 to meet the circumference of the circle (O) in D, E, F, and PA_2, PB_2, PC_2 to meet the same circumference in G, H, K. It is pretty well known and may easily be proved (see Leybourn's *Mathematical Repository*, Vol. VI., part 2, art. 14), that A_1, B_1, C_1 are the middle points of PD, PE, PF, and A_2, B_2, C_2 the middle points of PG, PH, PK. Now, as A_1, B_1, C_1, A_2, B_2, C_2, A_3, B_3, C_3 are the middle points of nine segments of chords all radiating from the same point P, it follows by the lemma, that these nine points lie in the circumference of a circle whose centre is the middle point of OP, and its radius half the radius of the circle (O).

COR.—It is also evident by the lemma that all lines drawn from P to the circumference of (O) will be bisected by the circumference of the nine point circle. It was this property, which I first met with in Whitworth's *Modern Analytical Geometry*, that suggested the preceding unique and symmetrical solution.

2309. (Proposed by Professor CAYLEY.)—Show that for n things

$$1-(\text{no. of partitions into 2 parts}) + 1\,.\,2\,(\text{no. of partitions into 3 parts})$$
$$\ldots\ldots \pm 1.2.3\ldots(n-1)\,(\text{no. of partitions into } n \text{ parts}) = 0.$$

For instance, $n=4$; partitions of (a, b, c, d) into two parts are (a, bcd), (b, cda), (c, dab), (d, abc), (ab, cd), (ac, db), (ad, bc); no. is $=7$. Partitions into 3 parts are (ab, c, d), (ac, b, d), (ad, b, c), (bc, a, d), (b, d, ac), (cd, a, b); no. is $=6$. Partition into 4 parts is (a, b, c, d); no. is $=1$. And we have

$$1-1\,.\,7+2\,.\,6-6\,.\,1 = 13-13 = 0.$$

I. *Solution by the* PROPOSER.

Write $n = a\alpha + b\beta + c\gamma + \ldots$, where $\alpha, \beta, \gamma \ldots$ are positive integers all of them different, and $\alpha, \beta, \gamma \ldots$ are positive integers; and consider the partitions wherein we have a parts each of α things, b parts each of β things, &c. Writing as usual $\Pi(n) = 1\,.\,2\,.\,3\ldots n$, the number of partitions of the form in question is $= \dfrac{\Pi n}{\Pi a\,.\,\Pi b\ldots(\Pi\alpha)^a(\Pi\beta)^b\cdots}$; whence, putting for shortness $a+\beta+\ldots = p$, the theorem may be written $\Sigma(-)^{p-1}\dfrac{\Pi(p-1)\,\Pi n}{\Pi a\,.\,\Pi b\ldots(\Pi\alpha)^a(\Pi\beta)^b\cdots} = 0$, the summation extending to all the partitions $n = a\alpha + b\beta + \ldots$, as explained above.

Now if the n quantities $x, y, z\ldots$ are the nth roots of unity, we have $x+y+z\ldots = 0$, and therefore also $(x+y+z\ldots)^n = 0$, and the general term on the left-hand is $\dfrac{\Pi n}{(\Pi\alpha)^a(\Pi\beta)^b\cdots}[\alpha^a\beta^b\ldots]$, where $[\alpha^a\beta^b\ldots]$ denotes the symmetrical function $\Sigma x^\alpha y^\alpha\ldots(a \text{ factors})\, u^\beta v^\beta\ldots(b \text{ factors})\ldots$ of the roots $x, y, z, u, v\ldots$ of the equation $\theta^n-1=0$; where, as above, $n = a\alpha + b\beta + \ldots$ Now by a formula not, I believe, generally known, but which is given on p. 175 of the translation of Hirsch's Algebra (Hirsch's *Collection of Examples*,

&c. on the Literal Calculus and Algebra, translated by the Rev. J. A. Ross, London, 1827), the value of the sum in question is $= (-)^{p-1}\dfrac{\Pi(p-1)}{\Pi a\,.\,\Pi b\,\ldots}\,n$, where $p = a+b+\ldots$, (the sign $\pm$, given in the formula as quoted, is at once seen to be $(-)^{p-1}$); whence, substituting and omitting the factor n, we have

$$\Sigma\,(-)^{p-1}\,\frac{\Pi\,(p-1)\,\Pi n}{\Pi a\,.\,\Pi b\,\ldots\,(\Pi\alpha)^{a}\,(\Pi\beta)^{b}\ldots} = 0,$$

which is the required theorem.

OBSERVATION.—In Cauchy's *Exercises d'Analyse, &c.*, tom. III., p. 173, is given a formula relating to the same mode of partition of the number n, viz. this is

$$\Sigma\,\frac{\Pi n}{\Pi a\,.\,\Pi b\ldots a^{a}\,\beta^{b}\ldots} = \Pi n.$$

I have somewhere made the remark that, on the left hand side, the terms which belong to the odd and the even values of $a+b+\ldots\ (=p)$ are equal, and that we have therefore $\Sigma\,(-)^{p-1}\,\dfrac{\Pi n}{\Pi a\,.\,\Pi b\ldots a^{a}\,\beta^{b}\ldots} = 0$, which is a theorem having a curious analogy with that demonstrated above.

II. *Solution by* SAMUEL ROBERTS, M.A.

The number of partitions of n things into p parcels with permutations of the parcels is given by the coefficient of $x_1x_2\ldots x_n$ in $\{\Sigma x_1 + \Sigma x_1 x_2 + \&c.\}$ or $P(1+x_1)-1$. We have therefore for the formula of the question

$$\text{Co. of } x_1x_2\ldots x_n \text{ in } \frac{P(1+x_1)-1}{1} - \frac{\{P(1+x_1)-1\}^2}{2} + \frac{\{P(1+x_1)-1\}^3}{3} - \&c.$$

But we may represent this series by $\log\{1+P(1+x_1)-1\}$ or $\log\{P(1+x_1)\}$ or $\Sigma\log(1+x_1)$, which expression when developed contains no term whose argument is $x_1x_2\ldots x_n$.

1912. (Proposed by Professor MANNHEIM.)—If a and b be the points of contact, with a curve of the third class, of a double tangent; and if this tangent be intersected in m, n, p by the three tangents to the curve which can be drawn from any point in the plane, then $\dfrac{am\,.\,an\,.\,ap}{bm\,.\,bn\,.\,bp} = \dfrac{\rho_a}{\rho_b}$, where ρ_a, ρ_b are the radii of curvature at the points a and b.

Solution by PROFESSOR CREMONA.

M. MANNHEIM a démontré (Poncelet, *Appl. d'Analyse et de Géom., tom. II., p.* 165) que, si l'on désigne par ρ_A, ρ_B les rayons de courbure au point double d d'une courbe de 3e ordre on a

$$\frac{\rho_A}{\rho_B} = \frac{\text{sen}\,(B, M)\,.\,\text{sen}\,(B, N)\,.\,\text{sen}\,(B, P)}{\text{sen}\,(A, M)\,.\,\text{sen}\,(A, N)\,.\,\text{sen}\,(A, P)},$$

où A, B sont les deux tangentes au point double, et M, N, P représentent les directions asymptotiques.

Transformons ce théorème par les polaires réciproques. On aura une courbe de 3^e classe avec une tangente double D, soient a, b les points de contact; o le centre du cercle directeur; m, n, p les points où D est coupé par les tangentes de la courbe, issues de o. L'angle (B, M) est égal à l'angle bom; donc

$$\frac{\text{sen (B. M)}}{\text{sen (A, M)}} = \frac{bm}{am} \cdot \frac{oa}{ob}, \text{ et par suite } \frac{\rho_A}{\rho_B} = \frac{bm \,.\, bn \,.\, bp}{am \,.\, an \,.\, ap} \cdot \left(\frac{oa}{ob}\right)^3.$$

Or M. Mannheim a démontré aussi (*Journal de M. Liouville*, tom. XI.) la relation $\frac{1}{\rho_a \cdot \rho_A} = \text{sen}^3\,(oa, \text{D})$, $\frac{1}{\rho_b \cdot \rho_B} = \text{sen}^3\,(ob, \text{D})$,

où ρ_a, ρ_b sont les rayons de courbure de la courbe de 3^e classe aux points a, b. Et, comme on a évidemment $\frac{oa}{ob} = \frac{\text{sen }(ob,\text{ D})}{\text{sen }(oa,\text{ D})}$, il s'ensuit que

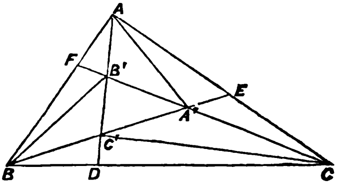
$$\frac{\rho_a}{\rho_b} = \frac{am \,.\, an \,.\, ap}{bm \,.\, bn \,.\, bp}.$$

2338. (Proposed by H. M. Taylor, B.A.) — ABC is a triangle D, E, F are points in the sides BC, CA, AB respectively, such that

$$\text{BD : DC = CE : EA = AF : FB} = l : m.$$

Find the area of the triangle formed by the three lines AD, BE, CF.

I. *Solution by* T. J. Sanderson, B.A.



Put $\frac{l}{m} = k$; and denote the triangles ABC, A'B'C' by Δ, Δ', and the sum of the 3 triangles AB'F, BC'D, CA'E by L. Then we have triangle ABD = k . ADC, &c.; therefore, by addition, we get

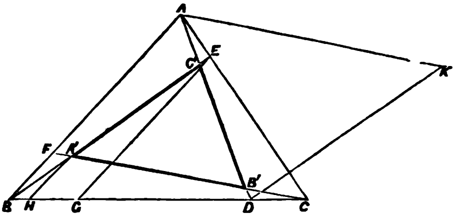
$\Delta - \Delta' + \text{L} = k\,(2\Delta + \Delta' - \text{L})$, or $\Delta\,(1 - 2k) - \Delta'\,(1 + k) + \text{L}(1 + k) = 0$ .. (1).

Join AA', BB', CC'; then triangle B'BD = k . B'DC, BB'F = $\frac{1}{k}$.AB'F,&c.; therefore, by addition, we get

$\Delta - \Delta' = k\,(\Delta + 2\Delta' - \text{L}) + \frac{\text{L}}{k}$, or $\Delta(1 - k)\,k - \Delta'(1 + 2k)k + (k^2 - 1)\text{L} = 0$...(2).

Eliminating L from (1) and (2), we have $\Delta = \frac{k^2 - 2k + 1}{k^2 + k + 1} \,.\, \Delta$.

II. *Solution by* Stephen Watson.



Let A′B′C′ be the triangle formed by the lines AD, BE, CF, and draw EG, A′H parallel to AB. Then obviously EG = AF, BG = CD; therefore by similar triangles we have

$$\text{EG or AF} : \text{BG or DC} = \text{A}'\text{H} : \text{BH},$$

and $$\text{CB} : \text{BF} = \text{BH} : \text{BF} - \text{A}'\text{H},$$

therefore $$\text{CB} \,.\, \text{AF} : \text{DC} \,.\, \text{BF} = \text{A}'\text{H} : \text{BF} - \text{A}'\text{H},$$

therefore $$\text{CB} \,.\, \text{AF} : \text{CB} \,.\, \text{AF} + \text{DC} \,.\, \text{BF} = \text{A}'\text{H} : \text{BF},$$

whence $$\text{A}'\text{H} = \frac{\text{CB} \,.\, \text{AF} \,.\, \text{BF}}{\text{CB} \,.\, \text{AF} + \text{DC} \,.\, \text{BF}} = \frac{\text{AB} \,.\, lm}{l(l+m)+m^2}.$$

Denoting the triangle ABC by Δ, we have

$$\frac{\text{triangle BA}'\text{C}}{\Delta} = \frac{\text{perp. from A}' \text{ on BC}}{\text{perp. from A on BC}} = \frac{\text{A}'\text{H}}{\text{AB}} = \frac{lm}{l^2+lm+m^2},$$

$$= \frac{\text{triangle CB}'\text{A}}{\Delta} = \frac{\text{triangle AC}'\text{B}}{\Delta};$$

therefore $$\text{triangle A}'\text{B}'\text{C}' = \left(1 - \frac{3lm}{l^2+lm+m^2}\right)\Delta = \frac{l^2-2lm+m^2}{l^2+lm+m^2}\Delta.$$

Cor. 1.—The triangles BA′C, CB′A, AC′B are equal in area; and the same may be shown of the triangles CA′A, AB′B, BC′C; and likewise of the triangles AA′B, BB′C, CC′A.

Cor. 2.—When $l = m$, the triangle A′B′C′ vanishes, as it ought, since then AD, BE, CF are concurrent.

III. *Solution by* J. Dale; *the* Rev. J. L. Kitchin, B.A.; *and others.*

1. Suppose this to mean the triangle whose vertices are the intersections of BE, CF; CF, AD; AD, BE. Let A′, B′, C′ represent these vertices, then the trilinear coordinates of these points are given by the equations

$$(\text{A}')\ldots\ \frac{ax}{lm} = \frac{by}{l^2} = \frac{cz}{m^2} = \frac{2\Delta}{l^2+lm+m^2}$$

$$(\text{B}')\ldots\ \frac{ax}{m^2} = \frac{by}{lm} = \frac{cz}{l^2} = \frac{2\Delta}{l^2+lm+m^2}$$

$$(\text{C}')\ldots\ \frac{ax}{l^2} = \frac{by}{m^2} = \frac{cz}{lm} = \frac{2\Delta}{l+lm+m^2}$$

and substituting these values in the expression for the area of a triangle in terms of the coordinates of its vertices, we get

$$\Delta\,\mathrm{A'B'C'} = \frac{8\Delta^3}{(l^2+lm+m^2)^3}\cdot\frac{\mathrm{R}}{2\Delta\,.\,abc}\begin{vmatrix} lm, & l^2, & m^2 \\ m^2, & lm, & l^2 \\ l^2, & m^2, & lm \end{vmatrix}$$

$$= \frac{(l^3-m^3)^2\,\Delta}{(l^2+lm+m^2)^3} = \frac{(l-m)^3\,\Delta}{l^3-m^3} = \frac{(l-m)^2\,\Delta}{l^2+lm+m^2}.$$

2. Suppose the question to be, to find the area of a triangle whose sides are respectively equal to AD, BE, CF. Draw from D a straight line parallel to BE, and from A a parallel to CF, intersecting the first parallel in K. Then the triangle ADK has its sides respectively equal to AD, BE, CF, and the coordinates of its angular points are as follows:—

$$(\mathrm{A})\ldots\ \frac{ax}{2\Delta} = \frac{y}{0} = \frac{z}{0} = 1,$$

$$(\mathrm{D})\ldots\ \frac{x}{0} = \frac{y}{mc} = \frac{z}{lb} = \frac{2\Delta}{(l+m)\,bc},$$

$$(\mathrm{K})\ldots\ \frac{ax}{l\,(l+m)} = \frac{by}{-l\,(l+m)} = \frac{cz}{(l+m)^2} = \frac{2\Delta}{(l+m)^2};$$

whence we find $\Delta\,\mathrm{ADK} = \dfrac{l^2+lm+m^2}{l^2+2lm+m^2}\,\Delta,$

and $\Delta\,\mathrm{ADK} : \Delta\,\mathrm{A'B'C'} = (l^2+lm+m^2)^2 : (l^2-m^2)^2.$

[When D, E, F are the *middles* of the sides, we have $l = m$ and $\Delta\,\mathrm{ADK} = \frac{3}{4}\Delta$. We thus obtain a proof of the following theorem, proposed in the *Giornale di Matematiche* for October, 1866 :—" Essendo dato un triangolo ABC, si formi colle mediane un secondo triangolo, dimostrare ; che l'area del triangolo, che ha per lati le mediane, ha un rapporto costante coll' area del triangolo ABC."]

2363. (Proposed by the Rev. J. Blissard.)—Prove that, if n be odd and >1,

$$\left\{\frac{\Delta}{1} - \frac{r+1}{2r+1}\cdot\frac{\Delta}{2} + \frac{(r+1)\,(r+2)}{(2r+1)\,(2r+2)}\cdot\frac{\Delta^3}{3} - \&\text{c.}\right\}0^n = 0.$$

Solution by the Proposer.

In Vol. III. of the *Reprint*, p. 87, it is proved that

$$\log\,(1+x) = \left\{\frac{m}{m+n}\ \frac{x}{1} - \frac{m\,(m+1)}{(m+n)\,(m+n+1)}\cdot\frac{x^2}{2} + \ldots\right\}$$

$$+\left\{\frac{n}{m+n}\cdot\frac{x}{1+x} + \frac{1}{2}\cdot\frac{n\,(n+1)}{(m+n)\,(m+n+1)}\cdot\left(\frac{x}{1+x}\right)^2 + \&\text{c.}\right\}.$$

Let m and n be each equal to r, then we have

$$\log\,(1+x) = \frac{1}{2}\left\{\frac{x}{1} - \frac{r+1}{2r+1}\cdot\frac{x^2}{2} + \&\text{c.}\right\} + \frac{1}{2}\left\{\frac{x}{1+x} + \frac{1}{2}\cdot\frac{r+1}{2r+1} + \&\text{c.}\right\}..(a).$$

Since this equation holds good quite generally, we may, on the principles of of *Representative Notation*, substitute for x any Representative Quantity or any function of such quantity.

Also since $\Delta^m 0^n$, which $= m^n - \frac{m}{1}\cdot(m-1)^n + \frac{m(m-1)}{1\,.\,2}(m-2)^n$, &c., may, when n is constant, be regarded as a function of m, the Δ here used is, in fact, a Representative Quantity, so that we may in the above expression substitute Δ for x, supplying the adjunct 0^n.

Hence
$$\log(1+\Delta)\,0^n = \frac{1}{2}\left\{\frac{\Delta}{1} - \frac{r+1}{2r+1}\cdot\frac{\Delta^2}{2} + \&c.\right\} 0^n$$
$$+ \frac{1}{2}\left\{\frac{\Delta}{1+\Delta} + \frac{1}{2}\left(\frac{\Delta}{1+\Delta}\right)^2 + \&c.\right\} 0^n.$$

But $\log(1+\Delta)\,0^n = 0\ (n>1)$ and $\left(\frac{\Delta}{1+\Delta}\right)^m 0^n = (-1)^{m+n}\,.\,\Delta^m 0^n$;

$$\therefore\ 0 = \left\{1+(-1)^{n+1}\right\}\left(\frac{\Delta}{1} - \frac{r+1}{2r+1}\cdot\frac{\Delta^2}{2} + \frac{(r+1)(r+2)}{(2r+1)(2r+2)}\cdot\frac{\Delta^3}{3} - \&c.\right)0^n.$$

Hence (n odd and >1), $\left(\frac{\Delta}{1} - \frac{r+1}{2r+1}\cdot\frac{\Delta^2}{2}\ \&c.\right)0^n = 0$ ....(*b*).

Cor.—In (*b*) put $2n+1$ for n, and multiply every term by $2r+1$, then
$$\left\{(2r+1)\frac{\Delta}{1} - (r+1)\frac{\Delta^2}{2} + \frac{(r+1)(r+2)}{2r+2}\frac{\Delta^3}{3} - \frac{(r+1)(r+2)(r+3)}{(2r+2)(2r+3)}\frac{\Delta^4}{4} + \&c.\right\} 0^{2n+1} = 0.$$

Now let $2r+1=0$, *i.e.*, $r=-\frac{1}{2}$, then casting out the factor $r+1$ or $\frac{1}{2}$, we get
$$\left\{\frac{\Delta^2}{2} - \frac{3}{2}\cdot\frac{\Delta^3}{3} + \frac{3\,.\,5}{2\,.\,4}\cdot\frac{\Delta^4}{4} - \&c.\right\} 0^{2n+1} = 0.$$

Similarly, by multiplying every term in (*b*) by $2r+2m+1$, and then making $r = -(m+\frac{1}{2})$ we obtain, after reduction,
$$\left\{\frac{\Delta^{2m}}{2m} - \frac{2m+1}{2}\cdot\frac{\Delta^{2m+1}}{2m+1} + \frac{(2m+1)\,2m+3)}{2\,.\,4}\cdot\frac{\Delta^{2m+2}}{2m+2} - \&c.\right\} 0^{2n+1} = 0.$$

1992. (Proposed by the Rev. R. Townsend, M.A.)—Construct geometrically (1) the four chords of contact with a given conic of the four inscribed conics which pass through three given points, (2) the eight planes of contact with a given quadric of the eight inscribed quadrics which pass through four given points.

Solution by the Proposer.

In the case of (1), if a, b, c be the three sides of the triangle determined by the three given points; a', b', c' the three chords intercepted upon them by the given conic; and A and A′, B and B′, C and C′ the three pairs of points which divide harmonically the three pairs of segments a and a', b and b', c and c', respectively; then, the six latter points lie three by three on four lines, viz., ABC, B′C′A, C′A′B, A′B′C′, which are the four chords required.

In the case of (2), if a and d, b and e, c and f be the three pairs of opposite edges of the tetrahedron determined by the four given points; a' and d', b' and e', c' and f' the three pairs of opposite chords intercepted upon them by the given quadric; and A and A′, B and B′, C and C′, D and D′, E and E′, F and F′ the six pairs of points which divide harmonically the six pairs of segments a and a', b and b', c and c', d and d', e and e', f and f', respectively; then, the twelve latter points lie six by six on eight planes, viz.,

ABCDEF, ABCD′E′F′, B′C′AEFD′, C′A′BFDE′,
A′B′CDEF′, B′C′AE′F′D, C′A′BF′D′E, A′B′CD′E′F,

which are the eight planes required.

For, if A be the point at which any one of the four lines or eight planes intersects with the line of any two of the collinear segments a and a'; then since A has evidently the same polar line or plane with respect to the original and to the corresponding inscribed conic or quadric, therefore it has the same harmonic conjugate A′ with respect to the two chords a and a' which they intercept on that line; and therefore, &c.

The same appears also from the equations of the four conics in the former case and of the eight quadrics in the latter case. For if, in the former case, the trilinear equation of the given conic, referred to the triangle determined by the three given points, be

$$S = a\alpha^2 + b\beta^2 + c\gamma^2 + 2d\beta\gamma + 2e\gamma\alpha + 2f\alpha\beta = 0,$$

then, those of the four inscribed conics being evidently $S = (\sqrt{a}\,.\,\alpha \pm \sqrt{b}\,.\,\beta \pm \sqrt{c}\,.\,\gamma)^2$, and those of the four chords of contact with S being consequently $\sqrt{a}\,.\,\alpha \pm \sqrt{b}\,.\,\beta \pm \sqrt{c}\,.\,\gamma = 0$, therefore, &c., as regards (1).

And if, in the latter case, the quadriplanar equation of the given quadric referred to the tetrahedron determined by the four given points, be

$$S = a\alpha^2 + b\beta^2 + c\gamma^2 + d\delta^2 + 2l\beta\gamma + 2m\gamma\alpha + 2n\alpha\beta + 2p\alpha\delta + 2q\beta\delta + 2r\gamma\delta = 0,$$

then, those of the eight inscribed quadrics being evidently $S = (\sqrt{a}\,.\,\alpha \pm \sqrt{b}\,.\,\beta \pm \sqrt{c}\,.\,\gamma \pm \sqrt{d}\,.\,\delta)^2$, and those of their eight planes of contact with S being consequently $\sqrt{a}\,.\,\alpha \pm \sqrt{b}\,.\,\beta \pm \sqrt{c}\,.\,\gamma \pm \sqrt{d}\,.\,\delta = 0$, therefore, &c., as regards (2).

2299. (Proposed by the Rev. R. TOWNSEND, M.A.)—It is a known property, in curves of the second order, that, at any point of a conic, the chord of curvature, which coincides in direction with either of the two focal lines through the point, is equal in length to either of the two focal chords of the curve whose directions are parallel to the tangent at the point.

Show, analogously, in surfaces of the second order, that, at any point of a quadric, the chord of normal curvature, which coincides in direction with any one of the four bifocal lines through the point, is equal in length to any one of the four bifocal chords of the surface whose directions are parallel to the corresponding tangent at the point.

Solution by the PROPOSER.

If a be the primary semi-axis of the surface, p the perpendicular from the centre on the tangent plane at the point, θ the common angle between the normal and the four bifocal lines through the point, ρ the radius of curvature of the normal section determined by either opposite pair of bifocal lines, R the radius of the surface parallel to the corresponding tangent at the point, and C the common length of the four bifocal chords parallel to the same tangent; then, since (Salmon's *Geometry of Three Dimensions*, 2nd ed., Arts. 173, 175, and 186) $\cos\theta = \frac{p}{a}$, $C = \frac{2R^2}{a}$, $\rho = \frac{R^2}{p}$, therefore $2\rho\cos\theta = C$, and therefore, &c.

2223. (Proposed by J. GRIFFITHS, M.A.)—Through the vertices A, B, C of a triangle ABC any three parallel straight lines AX, BY, CZ are drawn, cutting the sides BC, CA, AB in X, Y, Z; show that the envelope of the axis of homology of the triangles ABC, XYZ is the maximum ellipse that can be inscribed in the triangle ABC.

I. *Solution by the* REV. R. TOWNSEND, F.R.S.

This property is a particular case of the following, viz., if a variable point move on a fixed line, its polar with respect to a fixed triangle envelopes the inscribed conic which with the fixed line divides the three sides of the triangle harmonically. This theorem may be proved as follows:—

If L be the fixed line, O the variable point, ABC the three vertices of the triangle, X, Y, Z the three points at which the three variable lines OA, OB, OC intersect with the opposite sides BC, CA, AB, and X′, Y′, Z′ the three at which the variable polar of the point O, or, which is the same thing, the axis of homology of the two triangles ABC and XYZ, intersects with the same sides; then, the three pairs of points X and X′, Y and Y′, Z and Z′ being harmonically conjugate to each other with respect to the three pairs B and C, C and A, A and B respectively, and the three systems of points X′, Y′, Z′ being consequently homographic with the three X, Y, Z, and therefore with the common system O, the line X′Y′Z′ consequently envelopes the conic which touches the three sides BC, CA, AB at the three points P′, Q′, R′ which, with the three P, Q, R at which they intersect with the line L, divide them harmonically.

The theorem may be otherwise proved as follows:—If α, β, γ be the coordinates of O, and x, y, z those of any point on its polar, both referred to the sides of the triangle ABC; then the equation of the polar being $\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 0$, when α, β, γ, as lying on L, are connected by an equation of the form $a\alpha + b\beta + c\gamma = 0$, that of its envelope is consequently $\sqrt{(ax)} + \sqrt{(by)} + \sqrt{(cz)} = 0$, which is that of a conic touching the three sides of ABC at their three harmonic conjugates with respect to their intersections with L.

N.B.—The two reciprocal properties converse to the above and its reciprocal, viz.,

(*a*) If a variable line envelope a fixed conic, its pole with respect to any fixed circumscribed triangle moves on a fixed line, the axis of homology of the circumscribed and corresponding inscribed triangles, and

(a') If a variable point describe a fixed conic, its polar with respect to any fixed inscribed triangle turns round a fixed point, the centre of homology of the inscribed and corresponding circumscribed triangles;

have been established for the circle, in *Modern Geometry*, Chaps. XV. and XIX.; the methods employed are applicable to any conic as well.

II. *Solution by* J. Dale; W. Chadwick; *and others.*

Let any line through the angle A cut the base BC in X; then using tangential coordinates (x, y, z) the equation to the point X may be put in the form $\frac{y}{m}+\frac{z}{n}=0$; also the parallel line through B being BY,

$$\text{the equation to Y is } \frac{z}{m+n}-\frac{x}{m}=0,$$

$$\text{and the equation to Z is } \frac{x}{n}-\frac{y}{m+n}=0.$$

From these we find the coordinates of the lines YZ, ZX, XY to be, respecspectively,

$$\frac{x}{mn}=\frac{y}{m(m+n)}=\frac{z}{n(m+n)},\quad \frac{x}{mn}=\frac{y}{m(m+n)}=\frac{z}{-n(m+n)},$$

$$\frac{x}{mn}=\frac{y}{-m(m+n)}=\frac{z}{n(m+n)}.$$

Let X′, Y′, Z′ be the points in which the lines YZ, ZX, XY meet the lines BC, CA, AB; then the equations of X′, Y′, Z′ are, respectively,

$$ny-mz=0,\quad (m+n)\,x+mz=0,\quad (m+n)\,x+ny=0.$$

These points lie on the axis of homology, and from the equation of any two

we get $$\frac{x}{-mn}=\frac{y}{m(m+n)}=\frac{z}{n(m+n)}\ (=k,\ \text{suppose}),$$

whence $$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{k}\left\{\frac{1}{m+n}\left(\frac{1}{m}+\frac{1}{n}\right)-\frac{1}{mn}\right\}=0.$$

But $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=0$ is the equation of a conic inscribed in the triangle ABC, and touching the sides at their middle points, that is to say, it is the equation of the maximum ellipse that can be inscribed in the triangle.

2241. (Proposed by J. Griffiths, M.A.)—If C denote the circle which circumscribes any three given points α, β, γ; and T the four circles which touch the sides of the triangle formed by joining the points in question; prove that the two points common to C and T (*i. e.* the circular points at infinity) are the foci of the two imaginary parabolas which pass through the centres of T. Also show that the curve-locus of the foci of all parabolas which can be drawn through α, β, γ passes through the *ten* points where the circle C cuts the system T.

I. *Solution by the* PROPOSER.

1. To show that the foci of the two imaginary parabolas drawn through the centres of the four circles which touch the sides of a given triangle ABC coincide with the circular points at infinity.

The two parabolas which pass through the points in question are given by the equation $l\alpha^2+m\beta^2+n\gamma^2=0$,

where $$l+m+n=0, \text{ and } \frac{a^2}{l}+\frac{b^2}{m}+\frac{c^2}{n}=0;$$

also, if f, g, h denote the coordinates of one of the foci, we have

$$\frac{-af+bg+ch}{l}=\frac{af-bg+ch}{m}=\frac{af+bg-ch}{n}.$$

Hence it is easily seen that the foci lie on the straight line $af+bg+ch=0$ and on the nine-point circle of the triangle of reference; that is to say, they are the circular points at infinity.

2. In the same way it may be proved that the foci of the two parabolas which pass through the points $(a, b, c), (-a, b, c), (a, -b, c), (a, b, -c)$ lie on the straight line $\cos A\,.\,\alpha+\cos B\,.\,\beta+\cos C\,.\,\gamma=0$,

and circle $a\cos A\,.\,\alpha^2+b\cos B\,.\,\beta^2+c\cos C\,.\,\gamma^2-(ab\gamma+b\gamma\alpha+c\alpha\beta)=0$;

that is (see *Note on the Geometry of the Triangle*, Reprint, Vol. VI., p. 83) if A′, B′, C′ be the vertices of the triangle formed by the tangents at A, B, C to the circle (ABC), the two points common to the circles (ABC), (A B C′) are the foci of the two parabolas which pass through the points A′, B′, C′ and the centre of homology of the triangles ABC, A′B′C′. Hence the theorem in the second part of the question is easily seen to follow.

II. *Solution by* F. D. THOMSON, M.A.

It may be shown as on p. 338 of Salmon's *Conics*, that the foci of any parabola whose *tangential* equation is $\Sigma=0$, are given by the tangential equation $\Theta'\Sigma=\Delta\Sigma'$......(i), where $\Theta'=0$ is the condition that $\Sigma=0$ should be a rectangular hyperbola, $\Delta=0$ the discriminant of Σ, and $\Sigma'=0$ the tangential equation of the circular points at infinity. Now let O, O′, O″, O‴ be the centres of the system [T]; then taking the triangle as that of reference, the equation to a conic round OO′O″O‴ is of the form

$$(y-z)(y+z)=k(x^2-y^2), \text{ or } kx^2-(k+1)y^2+z^2=0....\text{(ii)}.$$

The tangential equation of the conic (ii) is therefore

$$\Sigma\equiv-(k+1)\lambda^2+k\mu^2-k(k+1)\nu^2=0.........,.........\text{(iii)},$$

also $$\Sigma'\equiv\lambda^2+\mu^2+\nu^2-2\mu\nu\cos A-B-2\nu\lambda\cos B-2\lambda\mu\cos C..\text{(iv)}.$$

Now $$\Delta\equiv k^2(k+1)^2, \text{ and } \Theta'\equiv-k^2(k+1)+k(k+1)^2-k(k+1)=0,$$

therefore the tangential equation to the foci becomes $0\,.\,\Sigma=k^2(k+1)^2\,\Sigma'$, hence the foci are both impossible, and coincide with the circular points.

In fact, any conic round OO′O″O‴ is a rectangular hyperbola.

2302. (Proposed by J. GRIFFITHS, M.A.) — Let A, B, C denote the vertices of a given triangle ABC; Q, R the opposite extremities of any diameter of the circle circumscribing it; then it is required to prove that the centres of the five equilateral hyperbolas which pass through A, B, C, Q, R, taken four and four together, all lie on the nine-point circle of the triangle.

Solution by J. Dale; H. Tomlinson; *and others.*

Since the centre of every equilateral hyperbola passing through A, B, C lies on the nine-point circle of ABC, it follows that the centres of the hyperbolas ABCQ and ABCR must lie on the nine-point circle. The centre of the hyperbola BCQR lies at the common point of intersection of the nine-point circles of each of the triangles BCQ, BCR, BQR, CQR, and this common point is the middle point of BC, which lies on the nine-point circle of ABC. So likewise the centres of the hyperbolas CAQR, ABQR, lie at the middle points of CA, AB respectively.

THEOREM ON THE COMPOSITION OF FORCES.

BY J. J. WALKER, M.A.

Assuming only that the resultant of two equal forces applied to a point is represented both in direction and magnitude by the diagonal of the rhombus under them when the forces are inclined at some one given angle 2θ, it is required to prove that the resultant of any two forces acting on a point is represented in magnitude by the diagonal of the parallelogram under them, when the forces are inclined at an angle θ.

DEMONSTRATION.

1. No principle is more evident than that the direction of the resultant of two forces acting at a given angle depends solely on the ratio of the forces, not on their absolute magnitude; as also does the magnitude of the resultant relatively to that of either of the given forces.

2. It may easily be shown to follow from the above principle, by employing the *ad absurdum* proof, that the angles which the resultant makes with its components being given, the ratio of the resultant to either component is given.

3. Suppose now P and Q to be the given forces and R to be their resultant; also let $P = mR$, and $Q = nR$. Consider P as the resultant of two forces P' and Q', the former acting in the direction R, the latter in a direction making with P, on the other side, the same angle which Q makes with R. Then (by 2) $P' = mP$, $Q' = nP$, or $P' = m^2R$, $Q' = mnR$.

Similarly consider Q as the resultant of two forces Q'', P'' acting, the former in the direction of R, the latter in a direction making the same angle, on the other side, with Q as P makes with R; then, as before, $P'' = mQ = mnR = Q'$ and $Q'' = nQ = n^2P$. Also the angle between P'' and Q' is plainly equal to 2θ, and is bisected by the direction of R. Hence the resultant of these two forces coincides in direction with R and is equal to $2mnR \cos\theta$, by the terms of the question.

Now it is evident that R must be equal to the resultant of the four forces P', Q'', P'', Q'; but the first two act in the same direction with R, as also the resultant of the last two: hence

$$R = P' + Q'' + 2mn\,R\cos\theta, \therefore R = m^2R + n^2R + 2mn\,R\cos\theta.$$

Multiplying by R, and observing that $m^2R^2 = P^2$, $n^2R^2 = Q^2$, $mnR^2 = PQ$, there results finally $R^2 = P^2 + Q^2 + 2PQ \cos \theta$.

It may be remarked that Laplace's proof of the magnitude of the resultant of two rectangular forces is the particular case of this theorem in which $\theta = 90^\circ$.

AT RANDOM: No. II. BY SEPTIMUS TEBAY, B.A.

On p. 65 of this volume of the *Reprint*, Mr. GODFRAY has offered some further observations on random lines, and by way of illustration has proposed the following problem:—"A line is drawn at random from the focus of an ellipse to the perimeter; what is the chance that its length is greater than the semi-major axis?" He adds three results, deduced from different *a priori* hypotheses, viz.:—Firstly, "A line whose chance of falling within the angle PSQ is proportional to that angle;" secondly, "A line whose chance of falling on the arc PQ is proportional to the length of the arc PQ;" thirdly, "A line whose chance of falling on the area PSQ is proportional to that area." And he remarks, "Here are three laws for a random line, leading to three different results, and I should be puzzled to say which of the three is the most natural."

Now with regard to a random line in general, there is no doubt but that it is one in which every position *a priori* is equally probable. A line may be drawn at random from a given point under the same conditions. The *first* of Mr. GODFRAY'S hypotheses is the only admissible one in the present case. The conditions of the problem are not that the line shall fall upon the arc PQ, nor on the area PSQ, but that it shall be drawn at random from the point S; and it matters not what be the length of the line considered so that it be drawn within the angular space which determines the probability required. If the second hypothesis were generally true, then between the points P, Q a line of infinite length may be supposed to lie within the angle PSQ, while the remaining portion of the curve is finite. In this case the chance would be certainty, a conclusion evidently absurd. The same remark applies to the third hypothesis.

AT RANDOM: No. III. *By* HUGH GODFRAY, M.A.

I would add a few words to my former communication on the necessity of strict definitions of "random" quantities, when such quantities are to be used as elements of a mathematical investigation.

It will be readily admitted, as has been clearly pointed out by Mr. CROFTON on p. 84 of this volume of the *Reprint*, that the first essential step, without which all numerical calculation is impossible, is to specify the assemblage of quantities, *all equally likely*, to which the random one belongs. This constitutes what I call the special law of the quantities. The law applies to the assemblage of quantities, and the fact that they are all equally probable shows that no further restriction is implied in the selection of the single one.

Objection has been taken to the phrase, "a random line drawn according to a special law." I admit that the wording is somewhat loose, but I can scarcely think it altogether unintelligible. As remarked by Mr. CROFTON, there is no contradiction in subjecting a random entity to law, provided in some respect it is left free. The sense I attached to it, and which I meant it to convey, was, "a line chosen at random among those, all equally probable, that can be drawn subject to a given law."

I must again refer to the example which I selected in a former paper:—"A line drawn at random from the focus of an ellipse to the perimeter." Mr. WOOLHOUSE, in his remarks upon it, says, that the second case should be called a line drawn to a random point in the periphery, and the third a line drawn to a random point within the area of the ellipse; but that the first case, depending on the angle at the focus, is that which naturally and truly represents a line drawn at random. I would, if classifying the cases as Mr. WOOLHOUSE has done, have called this first case, "a line drawn in a random direction, or making a random angle with some initial line; and as I see no reason for saying that a straight line is *more naturally* determined by passing through a given point and making a given angle with a given straight line, than by passing through two given points, I feel that I am still justified in asserting that, unless the word "random" is defined in the enunciation, that is, the law of the quantities from which the random one is taken, the problem will justly admit of various solutions.

Some years ago, the following question was proposed in the examination for the Mathematical Tripos:—"From a point in the circumference of a circular field a projectile is thrown at random with a given velocity, which is such that the diameter of the field is equal to the greatest range of the projectile; find the chance of its falling within the field." The question received two kinds of solution. Some of the candidates, and these were in a majority, assumed that all azimuths were equally probable, and all altitudes equally probable. Others took a hemisphere with the point as centre, and assumed that the chance of the direction of projection meeting any portion of the surface would be proportional to the area of that portion. The two results are essentially different; and the examiner, as a matter of course, admitted both, as he could not say that one supposition was more correct than the other, since the question had not sufficiently defined the meaning he meant to attach to the word "random."

2286. (Proposed by W. H. LAVERTY.)—If we have $(n-2)$ sets of n quantities each, $(\alpha_1, \alpha_2 \ldots \alpha_n)$, $(\beta_1, \beta_2 \ldots \beta_n)$, $\ldots(\lambda_1, \lambda_2 \ldots \lambda_n)$, connected with the n quantities $(r_1, r_2 \ldots r_n)$ by $\frac{1}{2}n(n-1)$ equations of which the type form is

$$(\alpha_k-\alpha_l)^2+(\beta_k-\beta_l)^2+\ldots(\lambda_k-\lambda_l)^2 = r_k^2+r_l^2;$$

then show that $\dfrac{1}{r_1^2}+\dfrac{1}{r_2^2}+\ldots+\dfrac{1}{r_n^2}=0$ and $\dfrac{P_1}{r_1^2}+\dfrac{P_2}{r_1^2}+\ldots+\dfrac{P_n}{r_n^2}=0$,

where P is any one of the quantities $\alpha, \beta, \gamma \ldots \lambda$.

I. *Solution by* PROFESSOR CAYLEY.

Consider the case $n=4$; we have between $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, $(\beta_1, \beta_2, \beta_3, \beta_4)$, (r_1, r_2, r_3, r_4) six equations, such as the equation

$$(\alpha_1-\alpha_2)^2+(\beta_1-\beta_2)^2 = r_1^2+r_2^2 \ldots\ldots (12);$$

and it is in effect required to show that these equations give

$$\frac{1}{r_1^2} : \frac{1}{r_2^2} : \frac{1}{r_3^2} : \frac{1}{r_4^2} = (234) : -(341) : (412) : -(123),$$

where
$$(123) = \begin{vmatrix} \alpha_1, & \beta_1, & 1 \\ \alpha_2, & \beta_2, & 1 \\ \alpha_3, & \beta_3, & 1 \end{vmatrix} \text{ \&c.,}$$

viz., considering (α_1, β_1), (α_2, β_2), (α_3, β_3), (α_4, β_4) as the rectangular coordinates of four points in a plane, then (123) is the area (taken with a proper sign) of the triangle formed by the points 1, 2, 3; and the like for (234) &c.

Combining the equations as follows,—

$$(12) + (34) - (13) - (24)$$

the r's disappear, and we have an equation

$$(\alpha_1 - \alpha_4)(\alpha_2 - \alpha_3) + (\beta_1 - \beta_4)(\beta_2 - \beta_3) = 0,$$

which shows that the lines 14 and 23 intersect at right angles; similarly the lines 12 and 34, and also the lines 13 and 24, intersect at right angles; or starting from the given points 1, 2, 3, the point 4 is the intersection of the perpendiculars let fall from the angles 1, 2, 3 of the triangle 123 on the opposite sides respectively.

Again combining the equations as follows,

$$(12) + (13) - (23),$$

we obtain
$$r_1^2 = (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3) + (\beta_1 - \beta_2)(\beta_1 - \beta_3).$$

The entire system of equations will remain unaltered if we pass from the original axes to any other system of rectangular axes; hence taking the axes of x in the sense from 1 to 2 along the line 12, $\beta_1 - \beta_2$ becomes $= 0$, and we have

$$\alpha_2 - \alpha_1 = 12, \quad \alpha_3 - \alpha_1 = 1\,(12, 34);$$

viz., $\alpha_2 - \alpha_1$ is the distance 12 of the points 1 and 2, $\alpha_3 - \alpha_1$ is the distance 1 (12, 34) of the point 1 from the point (12, 34) which is the intersection of the lines 12 and 34; we have therefore

$$r_1^2 = 12 \,.\, 1\,(12, 34).$$

But similarly
$$r_2^2 = 21 \,.\, 2\,(12, 34) = 12 \,.\, (12, 34)\,2$$

(since $21 = -12$ and $2\,(12, 34) = -(12, 34)\,2$). And we have therefore

$$r_1^2 : r_2^2 = 1\,(12, 34) : (12, 34)\,2, \quad \text{or} \quad \frac{1}{r_1^2} : \frac{1}{r_2^2} = (12, 34)\,2 : 1\,(12, 34).$$

Write
$$\lambda = \frac{(12, 34)\,2}{12}, \quad \mu = \frac{1\,(12, 34)}{12}$$

where 1 (12, 34) and (12, 34) 2 are as above the distances from 1 to (12, 34) and from (12, 34) to 2; and, in the denominators, 12 is the distance from 1 to 2; we have $\lambda + \mu = 1$; the coordinates of (12, 34) are $\lambda\alpha_1 + \mu\alpha_2$, $\lambda\beta_1 + \mu\beta_2$, and the values of λ, μ are obtained by writing $\lambda\alpha_1 + \mu\alpha_2$, $\lambda\beta_1 + \mu\beta_2$ $\lambda + \mu$ for $x, y, 1$ in the equations

$$\begin{vmatrix} x, & y, & 1 \\ \alpha_3, & \beta_3, & 1 \\ \alpha_4, & \beta_4, & 1 \end{vmatrix} = 0$$

of the line 34. Making this substitution, we find

$$\lambda\,(134)+\mu\,(234)=0,\ \text{where as above}\ (134)=\begin{vmatrix}\alpha_1 & \beta_1 & 1\\ \alpha_3 & \beta_3 & 1\\ \alpha_4 & \beta_4 & 1\end{vmatrix},\ \text{\&c.},$$

we have therefore $\lambda : \mu = (234) : -(134) = (234) : -(341)$,

or what is the same thing $(12, 34)\,2 : 1\,(12, 34) = (234) : -(341)$;

and consequently $\dfrac{1}{r_1^2} : \dfrac{1}{r_2^2} = (234) : -(341)$;

or completing the system by symmetry

$$\frac{1}{r_1^2} : \frac{1}{r_2^2} : \frac{1}{r_3^2} : \frac{1}{r_4^2} = (234) : -(341) : (412) : -(123),$$

which is the required result.

In the case $n = 5$, we have between $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$, $(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5)$, $(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5)$, $(r_1, r_2, r_3, r_4, r_5)$ ten equations such as the equation

$$(\alpha_1-\alpha_2)^2+(\beta_1-\beta_2)^2+(\gamma_1-\gamma_2)^2 = r_1^2+r_2^2 \ \ldots\ldots\ldots\ldots (12).$$

We obtain as before the equation

$$(\alpha_1-\alpha_4)(\alpha_2-\alpha_3)+(\beta_1-\beta_4)(\beta_2-\beta_3)+(\gamma_1-\gamma_4)(\gamma_2-\gamma_3)=0,$$

which, considering $(\alpha_1, \beta_1, \gamma_1)$ &c. as the rectangular coordinates of five points 1, 2, 3, 4, 5 in space, signifies that the line 14 is at right angles to the line 23; the five points are therefore such that the line joining any two of them is at right angles to the line joining any other two of them, whence also the line joining any two is at right angles to the plane through the remaining three points. (The points 1, 2, 3, 4 form a tetrahedron such that 12 and 34, also 13 and 42, also 14 and 23 are at right angles to each other, two of these conditions imply the third; and this being so, if a further condition be satisfied, the perpendiculars from 1, 2, 3, and 4 on the opposite faces respectively, will meet in a point 5, and we shall have the system of points 1, 2, 3, 4, 5 related as above).

We further obtain as before

$$r_1^2=(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)+(\beta_1-\beta_2)(\beta_1-\beta_3)+(\gamma_1-\gamma_2)(\gamma_1-\gamma_3),$$

or taking the axis of x in the sense from 1 to 2 along the line 12, we have $\beta_1-\beta_2=0$, $\gamma_1-\gamma_2=0$, and the equation becomes

$$r_1^2=12\,.\,1\,(12, 345),\ \text{and similarly}\ r_2^2=12\,.\,(12, 345)\,2\,;$$

whence $\dfrac{1}{r_1^2} : \dfrac{1}{r_2^2} = (12, 345)\,2 : (12, 345)\,1.$

Writing then $\lambda = \dfrac{(12, 345)\,2}{12}$, $\mu = \dfrac{1\,(12, 345)}{12}$ (and therefore $\lambda+\mu=1$) we find (λ, μ) by substituting $\lambda\alpha_1+\mu\alpha_2$, $\lambda\beta_1+\mu\beta_2$, $\lambda\gamma_1+\mu\gamma_2$, $\lambda+\mu$ for $x, y, z, 1$ in the equation

$$\begin{vmatrix} x, & y, & z, & 1\\ \alpha_3, & \beta_3, & \gamma_3, & 1\\ \alpha_4, & \beta_4, & \gamma_4, & 1\\ \alpha_5, & \beta_5, & \gamma_5, & 1\end{vmatrix}=0$$

of the plane 345; we have thus

$\lambda\,(1345)+\mu\,(2345)=0$, that is $\lambda : \mu = (2345) : -(1345) = (2345) : (3451)$

whence $$(12, 345)\,2 : 1\,(12, 345) = (2345) : (3451),$$

that is $$\frac{1}{r_1^2} : \frac{1}{r_2^2} = (2345) : (3451),$$

or completing by symmetry

$$\frac{1}{r_1^2} : \frac{1}{r_2^2} : \frac{1}{r_3} : \frac{1}{r_4^2} : \frac{1}{r_5^2} = (2345) : (3451) : (4512) : (5123) : (1234),$$

which is the theorem for the case $n = 5$. The general case depends, it is clear, upon similar reasoning in a $(n-2)$ dimensional geometry; leading to the conception in this geometry of a figure of $(n-1)$ points such that the line joining any two of them is at right angles to the line joining any other two of them.

II. *Solution by the* PROPOSER.

LEMMA.—If we denote the determinant

$$\begin{vmatrix} 1 & 1 & \ldots\ldots & 1 & 1 & \ldots\ldots & 1 \\ \alpha_{p-1} & \alpha_p & \ldots\ldots & \alpha_n & \alpha_1 & \ldots\ldots & \alpha_{p-3} \\ \beta_{p-1} & \beta_p & \ldots\ldots & \beta_n & \beta_1 & \ldots\ldots & \beta_{p-3} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \lambda_{p-1} & \lambda_p & \ldots\ldots & \lambda_n & \lambda_1 & \ldots\ldots & \lambda_{p-3} \end{vmatrix}$$

by $(p-2)$, then evidently we have

$(1) \pm (2) + (3) \pm \ldots \pm (n) = 0$; and $P_1(1) \pm P_2(2) + P_3(3) \pm \ldots \pm P_n(n) = 0$.

Also, if in $(p-2)$ we change $p-1$ into $p-2$ without altering the other columns, we get $\pm (p-1)$ according as n is even or odd.

Now, taking three of the given conditions

$$(\alpha_k - \alpha_{k-1})^2 + C = r_k^2 + r_{k-1}^2; \quad (\alpha_k - \alpha_m)^2 + C = r_k^2 + r_m^2;$$
$$(\alpha_{k-1} - \alpha_m)^2 + C = r_{k-1}^2 + r_m^2;$$

where C represents similar terms involving $\beta, \gamma, \ldots \lambda$; we get, adding the first two and subtracting the last,

$$(\alpha_k - \alpha_{k-1})(\alpha_k - \alpha_m) + C = r_k^2,$$

where m may have any value from 1 to n, except k and $k-1$; and hence we have $n-2$ equations, involving $n-2$ ratios, whose numerators are $\alpha_k - \alpha_{k-1}$, $\beta_k - \beta_{k-1}, \ldots . \lambda_k - \lambda_{k-1}$, and denominators r_k^2. Solving we get

$$\frac{\alpha_k - \alpha_{k-1}}{\begin{vmatrix} \beta_k - \beta_1, & \gamma_k - \gamma_1 & \ldots\ldots & 1 \\ \beta_k - \beta_2, & \gamma_k - \gamma_2 & \ldots\ldots & 1 \\ \vdots & \vdots & & \vdots \end{vmatrix}} = \&c. = \frac{-r_k^2}{\begin{vmatrix} \alpha_k - \alpha_1, & \beta_k - \beta_1 & \ldots\ldots & \lambda_k - \lambda_1 \\ \alpha_k - \alpha_2, & \beta_k - \beta_2 & \ldots\ldots & \lambda_k - \lambda_2 \\ \vdots & \vdots & & \vdots \end{vmatrix}}$$

which will be found to reduce to

$$\frac{\alpha_k-\alpha_{k-1}}{Q}=\frac{\beta_k-\beta_{k-1}}{R}=\&c.=\frac{\pm r_k^2}{(k-1)},$$

where Q, R .... do not involve either k or $k-1$, and will remain unchanged when we interchange those two numbers. Interchanging then, and changing or not changing the sign of $(k-1)$ according to the lemma, we have

$$-\frac{\alpha_{k-1}-\alpha_k}{Q}=-\frac{\beta_{k-1}-\beta_k}{R}=-\&c.=\frac{\pm r_{k-1}^2}{(k)};$$

therefore $$\pm r_k^2(k)=\pm r_{k-1}^2(k-1).$$

Giving different values to k, we have

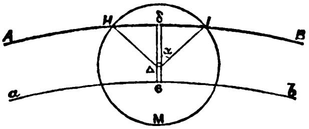
$$\frac{(1)}{\frac{1}{r^2}}=\frac{(\pm 2)}{\frac{1}{r_2^2}}=\frac{(3)}{\frac{1}{r_3^2}}=\&c.=\frac{\pm(n)}{\frac{1}{r_n^2}},$$

whence, by the lemma, &c.

1880. (Proposed by M. W. Crofton, B.A.)—If two points are taken at random within any closed convex contour, the probability (p) of their distance not exceeding a given length (ρ), which is small compared with the dimensions of the area, is approximately $p=\frac{\pi}{\Omega}\rho^2-\frac{2L}{3\Omega^2}\rho^3$, where L = length of contour, and Ω = enclosed area.

2. If the points are taken within a closed convex *surface* (S) inclosing a volume (V), we shall have $p=\frac{4\pi}{3V}\rho^3-\frac{\pi S}{4V^2}\rho^4$.

Solution by the Proposer.



1. Let AB be a portion of the bounding contour, and *ab* a *parallel* contour, at a normal distance ρ from AB. If Ω be the area enclosed by the whole contour AB, and ω that within *ab*, we shall have $\omega=\Omega-L\rho$; neglecting quantities of the second order.

Now if the first point (P) be taken within ω, and the other (Q) anywhere within Ω, the measure of the favourable cases is evidently $\pi\rho^2\omega$.

We have now to consider the case when P is taken within the band $\Omega-\omega$. Take an element δ of the contour, draw normals at its extremities, let Δ be an element of the area they enclose; from Δ as centre draw a circle, radius ρ. If P be taken within Δ, the measure of favourable cases is Δ. (segment HIM),

therefore $$dF=\rho^2(\pi-\alpha+\sin\alpha\cos\alpha)\,\delta\,.\,d\,.\,\rho\cos\alpha,$$

neglecting quantities of the second order, and considering the contour as a straight line in the vicinity of δ. Hence the number of favourable cases obtained by taking P within the space $\delta\epsilon$ will be

$$F = \rho^3\delta\int_0^{\frac{1}{2}\pi}(\pi - \alpha + \sin\alpha\cos\alpha)\sin\alpha\, d\alpha = \rho^3\delta\,(\pi - \tfrac{2}{3});$$

and if P be taken anywhere on the band $\Omega - \omega$, we have $F = \rho^3 L\,(\pi - \frac{2}{3})$. Hence the total of the favourable cases is

$$\begin{aligned} F &= \pi\rho^2\omega + \pi\rho^3 L - \tfrac{2}{3}\rho^3 L \\ &= \pi\rho^2(\Omega - L\rho) + \pi\rho^3 L - \tfrac{2}{3}\rho^3 L = \pi\rho^2\Omega - \tfrac{2}{3}\rho^3 L; \end{aligned}$$

therefore, as Ω^2 represents the whole number of cases, we have

$$p = \frac{\pi}{\Omega}\rho^2 - \frac{2L}{3\Omega^2}\rho^3.$$

[Hence it appears that, if the area be given, the probability is greatest when the figure is a circle.]

2. This expression may be obtained by a precisely analogous process.

1903. (Proposed by W. S. BURNSIDE, M.A.)—If normals drawn to a conic at the points A, B, C, D pass through a point O, and meet the curve again in the points A_1, B_1, C_1, D_1; prove that the five points O, A_1, B_1, C_1, D_1 lie on an equilateral hyperbola, touching the equilateral hyperbola through ABCD at O.

I. *Solution by the* PROPOSER.

First, to find the equation of the four normals: the equation of the lines joining the point (α, β, γ) to the intersections of two conics U and V is

$(UV_1 + U_1V - 2PQ)^2 = 4(UU_1 - P^2)(VV_1 - Q^2)$, (Salmon's *Conics*, p. 293).

In this case we have

$$U = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1, \quad P = \frac{\alpha x}{a^2} + \frac{\beta y}{b^2} - 1, \quad V_1 = 0,$$

$$V = 2(c^2xy + b^2\beta x - a^2\alpha y), \quad Q = a^2\beta x - b^2\alpha y - c^2\alpha\beta;$$

whence the above equation reduces to $U_1V^2 + 4Q(QU - PV) = 0$ for the equation of the normals. Putting $U = 0$ in the last equation, we get $V(VU_1 - 4PQ) = 0$. But as the points A, B, C, D lie on the curve V, the points A_1, B_1, C_1, D_1 lie on $VU_1 - 4PQ = 0$, which is an equilateral hyperbola; and it plainly touches V at O, $Q = 0$ being the equation of the tangent.

II. *Solution by* J. DALE; E. MCCORMICK; *and others.*

If through any point O in the plane of a conic four chords be drawn cutting the curve in the points A, A_1; B, B_1; C, C_1; D, D_1; then it may be shown that the conic through the five points O, A, B, C, D is similar to the conic through the five points O, A_1, B_1, C_1, D_1; but if the points A, B, C, D are the feet of the normals from O, then it is well known that OABCD is an equilateral hyperbola; therefore the conic $OA_1B_1C_1D_1$ is also an equilateral hyperbola.

To show that these hyperbolas touch in O, project the original conic into a circle having the projection of O for centre, then the chords AOA_1, BOB_1, COC_1, DOD_1 will be projected into diameters, and it is evident that the projections of the hyperbolas OABCD, $OA_1B_1C_1D_1$ will have a common tangent at the projection of the common point O_1: hence the conics OABCD, $OA_1B_1C_1D_1$ have a common tangent at O.

2246. (Proposed by H. R. GREER, B.A.)—Let U be a homogeneous expression in n (any number of) variables of the second degree: let A_n be the discriminant: erasing from A_n the right-hand column and bottom row, call the determinant so formed A_{n-1}: let this process and notation be continued until we arrive at A_1 (which will be the cofficient of some one of the squares of the variables in U): prove that if every one of these determinants, from A_n to A_1, be positive, U = 0 cannot be satisfied by any real values of the variables.

Solution by PROFESSOR SYLVESTER.

If we express U in the umbral notation by $(ax+by+cz+\ldots lt)^2$, I have shown long ago in the *Philosophical Magazine*, that

$$U = \begin{pmatrix} a \\ a \end{pmatrix} x^2 + \frac{\begin{pmatrix} ab \\ ab \end{pmatrix}}{\begin{pmatrix} a \\ a \end{pmatrix}} y_1^2 + \frac{\begin{pmatrix} abc \\ abc \end{pmatrix}}{\begin{pmatrix} ab \\ ab \end{pmatrix}} z_1^2 + \ldots\ldots + \frac{\begin{pmatrix} abc\ldots l \\ abc\ldots l \end{pmatrix}}{\begin{pmatrix} abc\ldots k \\ abc\ldots k \end{pmatrix}} t_1^2;$$

y_1, z_1, ... t_1, being real linear functions of x, y; x, y, z; .. ...; $x, y, z\ldots t$.

Hence if $\begin{pmatrix} a \\ a \end{pmatrix}$; $\begin{pmatrix} ab \\ ab \end{pmatrix}$; .... ; $\begin{pmatrix} abc\ldots l \\ abc\ldots l \end{pmatrix}$, (which are Mr. GREER'S minor coordinates), are all positive, U is incapable of a change of sign for real values of $x, y, z\ldots\ldots, t$; which proves the theorem.

2326. (Proposed by M. W. CROFTON, B.A.)—If three points be taken at random on the surface of a sphere, find the respective chances that the triangle they determine will have (1) all its angles *acute*, (2) all its angles *obtuse*.

Solution by EXHUMATUS.

1. Three points are taken at random on a sphere, required the chance that the triangle formed is acute-angled. Let A, B be two of the points, then

the third point C must fall outside of a closed curve, having AB for a diameter, which curve is the locus of the point C when the angle at C is a right angle. This curve is defined by the equation $\tan b = \tan c \cos A$. Let its area $= S$, and let the lune formed by the two great circles passing through A and B perpendicular to AB be denoted by L. The point C is thus restricted to the space $L-S$. Now we have

$$S = 2\int_0^{\frac{1}{2}\pi} \left(1 - \frac{\cos c}{\sqrt{(1-\sin^2 c \sin^2 A)}}\right) dA;$$

also, A remaining fixed, B may take any position within a distance $\frac{1}{2}\pi$ of A; hence, the total number of positions of B and C being $(4\pi)^2$, the chance of an acute-angled triangle is

$$\frac{1}{16\pi^2}\int_0^{\frac{1}{2}\pi} (L-S)\, 2\pi \sin c\, dc$$

$$= \frac{1}{4\pi}\int_0^{\frac{1}{2}\pi} \left\{c - \int_0^{\frac{1}{2}\pi} \left(1 - \frac{\cos c}{\sqrt{(1-\sin^2 c \sin^2 A)}}\right) dA\right\} \sin c\, dc.$$

But it may be shown that $\displaystyle\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} \frac{\sin c \cos c\, dc\, dA}{\sqrt{(1-\sin^2 c \sin^2 A)}} = 1.$

Hence the required chance is found to be $\dfrac{1}{2\pi} - \dfrac{1}{8}$.

2. Again, to find the chance that the angles will all exceed a right angle. In this case, let first A and B be within the distance $\frac{1}{2}\pi$ of each other, then C is restricted to a space S' equal to S and exactly opposite to it on the sphere; if $AB > \frac{1}{2}\pi$, then C is simply restricted to the lune L' equal and opposite to L. Hence the chance required is

$$\frac{1}{16\pi^2}.\int_0^{\frac{1}{2}\pi} S\,.\,2\pi \sin c\, dc + \frac{1}{16\pi^2}\int_{\frac{1}{2}\pi}^{\pi} 2c\,.\,2\pi \sin c\, dc = \frac{3}{8} - \frac{1}{2\pi}.$$

2331. (Proposed by Professor Cayley.)—Show that it is possible to find (X, Y, Z) linear functions of the trilinear coordinates (x, y, z) such that the equations $xX = yY = zZ$ may determine four given points.

Solution by Samuel Roberts, M.A.

Expressing X, Y, Z in terms of x, y, z, we have equations of the forms

$$x(Ax + By + Cz) + y(A'x + B'y + C'z) = 0,$$
$$y(A'x + B'y + C'z) + z(A''x + B''y + C''z) = 0,$$

which may be interpreted as the equations of two conics, the one passing through xy, and the other through yz. These are the only conditions implied by the forms of the equations. Hence we can determine ABC, $A'B'C'$, $A''B''C''$, so that the two conics may pass through four given points. Therefore, &c.

2365. (Proposed by H. M. TAYLOR, B.A.)—Into how many distinct parts is (1) an infinite line divided by n points, (2) infinite space of two dimensions by n straight lines, and (3) infinite space of three dimensions by n planes?

I. *Solution by* E. HILL, B.A.

1. n points divide an infinite line into $n+1$ parts.

2. Let $n-1$ lines divide a plane into $f(n-1)$ areas; then, another line being drawn, it is cut by the $n-1$ lines in $n-1$ points, which, by (1), divide it into n parts. Each of these parts lies in a distinct area, and divides it into two; therefore

$$f(n) = f(n-1) + n \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\alpha).$$

Also $f(0) = 1$, since there is one space if there are no lines; hence

$$f(n) = n + (n-1) + \ldots + 1 + 1 = \tfrac{1}{2}n(n+1) + 1.$$

3. Let $n-1$ planes divide space into $F(n-1)$ spaces; then, another plane being drawn, it will be cut by the $n-1$ planes in $n-1$ lines, which will mark out on it $f(n-1)$ areas. Each area lies in a distinct space, which it divides into two

$$\therefore\ F(n) = F(n-1) + f(n-1), \text{ or } F(n) - F(n-1) = \tfrac{1}{2}(n-1)n + 1 \ \ldots(\beta).$$

Also $F(0) = 1$, as before; hence we have

$$F(n) = \tfrac{1}{2}\Sigma\left\{n(n-1)\right\} + \Sigma(1) + 1 = \tfrac{1}{6}(n+1)n(n-1) + n + 1$$
$$= \tfrac{1}{6}(n+1)(n^2 - n + 6).$$

II. *Solution by the* PROPOSER.

1. In any straight line let n points be taken; the line is then divided into $n+1$ distinct parts.

2. In any plane let n straight lines be drawn; any one will intersect each of the others, and therefore be divided into n distinct parts; each of these parts will divide a portion of superficial space, which was single before, into two parts. The nth line therefore adds to the number of the parts n. The total number therefore of distinct parts into which a plane is divided by n straight lines is

$$n + (n-1) + (n-2) + \&c. = \tfrac{1}{2}n(n+1) + C.$$

Now, $C = 1$, for when there is no line, the plane is whole; therefore the number of parts $= \tfrac{1}{2}n(n+1) + 1$.

3. Let n planes be drawn; any one will intersect each of the others in a straight line, and will therefore by (2) be divided into $\tfrac{1}{2}(n-1)n + 1$ distinct parts, each of which parts will divide a portion of solid space, which was single before, into two parts. The nth plane therefore adds to the number of the parts $\tfrac{1}{2}(n-1)n + 1$.

The whole number of parts therefore into which solid space is divided by n planes is

$$\left\{\tfrac{1}{2}(n-1)n + 1\right\} + \left\{\tfrac{1}{2}(n-2)(n-1) + 1\right\} + \&c. =$$
$$\tfrac{1}{6}(n-1)n(n+1) + (n-1) + C.$$

Now, $C = 2$, as, when there is no plane, space is undivided; therefore the number of solid parts is

$$\tfrac{1}{6}(n+1)\,n\,(n-1)+(n+1) = \tfrac{1}{6}(n+1)(n^2-n+6).$$

N.B.—The method of Induction may be used, but is dangerous, as it is difficult to count the portions as n increases. The usual result so arrived at for (3) is 2^n.

2373. (Proposed by Dr. SALMON, F.R.S.)—What is the condition that a circle should cut a conic orthogonally? How many circles can be drawn through two points to cut a given conic orthogonally?

Solution by the PROPOSER.

It is required to find the condition that some one of the four tangents to the conic at the points where it is met by the circle shall pass through the centre of the circle.

Now the equation to the tangents to a conic U at the points where it is met by another conic V is, in terms of the invariants,

$$(\Theta U - \Delta V)^2 = 4\Delta U(\Theta' U - F).$$

In the above, we have only to write for V, $(x-\alpha)^2+(y-\beta)^2-r^2$; to form the invariants, and then substitute (α, β) for (x, y). The equation is of the sixth degree in the coefficients of U; therefore six conics can be drawn through four points orthogonal to the given circle.

1914. (Proposed by W. B. DAVIS, B.A.)—Required all the roots of the equation $x^7+28x^4 = 480$.

Solution by the PROPOSER.

Putting, as usual, i for $\sqrt{(-1)}$, the seven roots are

$r_1 = +1{\cdot}9228841.$

$r_2 = -2{\cdot}4580892.$

$r_3 = -2{\cdot}5778036.$

$r_4 = +1{\cdot}6843159+2{\cdot}6637914i.$

$r_5 = +1{\cdot}6843159-2{\cdot}6637914i.$

$r_6 = -0{\cdot}1278113+1{\cdot}9874234i.$

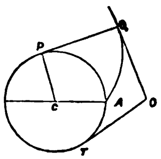
$r_7 = -0{\cdot}1278113-1{\cdot}9874234i.$

Developed to 42 decimals by Horner's method, the real positive root is

$$r_1 = 1{\cdot}922884153250909090149019026520588634529015.$$

2320. (Proposed by M. W. Crofton, B.A.)—Draw a circle and its involute, and find the locus of the centre of a circle which cuts both curves orthogonally.

Solution by James Dale.



Let P, Q be corresponding points in a circle and its involute, AC $= a$ and $\angle$ PCA $= \theta$; then the coordinates (X, Y) of Q are

$$X = a(\cos\theta + \theta\sin\theta),\quad Y = a(\sin\theta - \theta\cos\theta).$$

If O(x, y) be the centre of a circle cutting both curves orthogonally, then (1) O must lie in the tangent to the involute at the cutting point, and (2) the tangent to the circle from O must be equal to the tangent to the involute. The first condition gives

$$x\sin\theta - y\cos\theta = a\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and the second gives

$$x^2 + y^2 - a^2 = (X - x)^2 + (Y - y)^2 \ldots\ldots\ldots\ldots\ldots (2),$$

which on reduction becomes

$$Xx + Yy = a^2 + \tfrac{1}{2}a^2\theta^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

or $$x(\cos\theta + \theta\sin\theta) + y(\sin\theta - \theta\cos\theta) = a + \tfrac{1}{2}a\theta^2 \ldots\ldots\ldots (4),$$

or $$x\cos\theta + y\sin\theta = a - \tfrac{1}{2}a\theta^2 \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Adding the squares of (1) and (5), we get

$$x^2 + y^2 - a^2 = (\tfrac{1}{2}a\theta^2)^2 = s^2,$$

where s = arc of the involute from A to Q; therefore OT = OQ = AQ; consequently O lies on the involute of the involute, beginning at A, which is therefore the locus of all circles cutting both the curves orthogonally.

2210. (Proposed by S. Watson.)—A semi-ellipse is placed in a vertical position, and a heavy particle is just set in motion at the highest point of the curve, and descends down it by its own weight; required the point at which it will leave the curve, and its velocity at that point.

Solution by the Proposer.

Let (x, y) be the coordinates of the point where the particle leaves the curve; e the eccentricity of the ellipse, a and b its semi-axes, b being vertical; v the velocity, and r the radius of curvature at the point (x, y); and θ the angle which the normal at (x, y) makes with the axis of x.

Then $v^2 = 2g(b - y)$, and the centrifugal force at (x, y) is

$$= \frac{v^2}{r} = \frac{2g(b-y)a^4b^4}{(a^4y^2 + b^4x^2)^{\frac{3}{2}}},\ \text{and also} = g\sin\theta = \frac{ga^2y}{(a^4y^2 + b^4x^2)^{\frac{1}{2}}},$$

therefore $$\frac{2(b-y)\,a^2b^4}{a^4y^2+b^4x^2}=y, \text{ or } a^2e^2y^3+3b^4y-2b^5=0,$$

whence $$y=\frac{b^{\frac{5}{3}}}{ea^{\frac{2}{3}}}\left\{(1+e)^{\frac{1}{3}}-(1-e)^{\frac{1}{3}}\right\},$$

which determines the point (x, y), and the required velocity is then given by $v^2=2g(b-y)$.

2347. (Proposed by X. U. J.)—No real straight line can be drawn parallel to the imaginary straight line whose equation in trilinear coordinates is $$l\alpha+m\beta+n\gamma+i\,(l'\alpha+m'\beta+n'\gamma)=0, \text{ (where } i=\sqrt{-1}),$$

unless $$\begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ a, & b, & c \end{vmatrix}=0.$$

I. *Solution by* Samuel Roberts, M.A.

A real straight line meets the line at infinity in a real point. Therefore the one real point of the given imaginary line must be on the line at infinity, and the condition for this is the determinant expressed in the question.

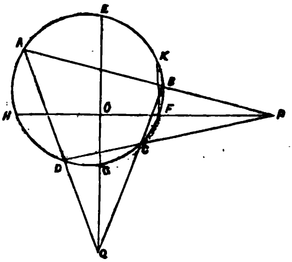
II. *Solution by* S. Watson; J. Walmsley; Rev. R. H. Wright, M.A.; Rev. J. L. Kitchin, M.A.; J. Dale; *and others.*

A parallel to the stated imaginary line is
$$l\alpha+m\beta+n\gamma+i\,(l'\alpha+m'\beta+n'\gamma)+p\,(a\alpha+b\beta+c\gamma)=0,$$
or $$(l+pa)\,\alpha+(m+pb)\,\beta+(n+pc)\,\gamma+i\,(l'\alpha+m'\beta+n'\gamma)=0\ldots\ldots(1),$$
and in order that this may be real, it is necessary that
$$l+pa : m+pb : n+pc = l' : m' : n' \ldots\ldots\ldots\ldots\ldots(2);$$
for when these conditions have place, (1) becomes, by omitting the factor $1+\dfrac{l'i}{l+pa}$, the real line $l'\alpha+m'\beta+n'\gamma=0.$

Now (2) is the same as
$$\begin{vmatrix} l', & m' \\ a, & b \end{vmatrix} p = \begin{vmatrix} l, & m \\ l', & m' \end{vmatrix} \text{ and } \begin{vmatrix} m', & n' \\ b, & c \end{vmatrix} p = \begin{vmatrix} m, & n \\ m', & n' \end{vmatrix};$$
whence, eliminating p, we have $$\begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ a, & b, & c \end{vmatrix}=0.$$

2385. (Proposed by T. T. Wilkinson, F.R.A.S.)—If the circumference of a circle be divided into any four arcs, and these arcs be bisected, prove that the chords joining the points of bisection intersect at right angles.

Solutions by Paul Branquart; G. A. Ogilvie; *the* Proposer; *and others.*



1. Let AB, BC, CD, DA be the four arcs, and E, F, G, H their middle points; then 2AE + 2AH + 2CF + 2CG = 360°,

$\therefore$ EH + FG = EF + HG = 90°.

Draw FK parallel to GE; then EK = FG, and HEK = HE + FG = 180°, therefore ∠HFK = HOE = an angle in a semicircle = a right angle.

2. *Otherwise:* Let the chords AB, DC meet in P; and AD, BC in Q; then PFH, QGE are the bisectors of the angles P and Q, and are therefore (Solutions to Quest. 1950, *Reprint*, Vol. V., p. 105) at right angles to each other.

1933. (Proposed by C. Taylor, M.A.)—If α, β be the eccentric angles of two adjacent vertices of a polygon inscribed in a fixed ellipse, and if the polygon envelope a fixed confocal, then will $\Sigma\cos(\alpha-\beta)$ = a constant.

Solution by the Proposer.

The length of the side (α, β) varies as $\sin^2\frac{1}{2}(\alpha-\beta)$ (Salmon's *Conics*), and it may be shown by Graves' Theorem, or otherwise, that (the number of sides being constant) the perimeter is constant. Hence each of the expressions $\Sigma\sin^2\frac{1}{2}(\alpha-\beta)$, $\Sigma\cos^2\frac{1}{2}(\alpha-\beta)$, $\Sigma\cos(\alpha-\beta)$ is constant.

2377. (Proposed by R. Tucker, M.A.)—If the distance between the foci of an ellipse subtend angles ϕ, ψ at the extremities of a pair of semiconjugate diameters, then

$$\tan^2\tfrac{1}{2}\phi + \tan^2\tfrac{1}{2}\psi = \frac{a^2e^2}{b^2}.$$

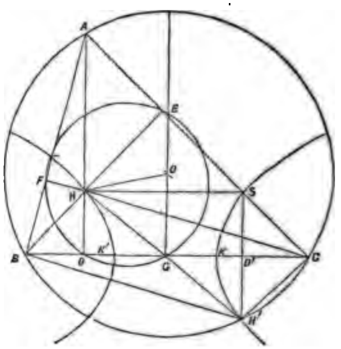
Solution by S. Watson; E. McCormick; T. J. Sanderson, B.A.; G. A. Ogilvie; *and others.*

Let F, f be the foci; P, Q the extremities of conjugate semi-axes, and PR the normal at P, bisecting the angle FPf. Then, if P be denoted by (x', y'), the ordinate of Q is $y_1 = \frac{bx'}{a}$; also $\tan \mathrm{PF}f = \frac{y'}{x'+c}$, $\tan \mathrm{PR}f = \frac{a^2y'}{b^2x'}$; hence $\tan\frac{1}{2}\phi = \tan(\mathrm{PR}f - \mathrm{PF}f) = \frac{cy'}{b^2}$. Similarly $\tan\frac{1}{2}\psi = \frac{cy_1}{b^3} = \frac{cx'}{ab}$;

therefore $$\tan^2\tfrac{1}{2}\phi + \tan^2\tfrac{1}{2}\psi = \frac{c^2}{b^2}.$$

1755. (Proposed by R. Tucker, M.A.)—Conics are described passing through two angular points of a triangle, and having one focus coinciding with the point of intersection of the perpendiculars from the angles on the opposite sides; show that the locus of the other focus is a conic.

Solution by the PROPOSER.



That the locus is a conic, has been shown on page 29 of this volume of the *Reprint.* I shall now proceed to give a fuller discussion of the question.

Take $\mathrm{AH} = \delta_1 = 2\mathrm{R}\cos\mathrm{A}$,
$\mathrm{BH} = \delta_2 = 2\mathrm{R}\cos\mathrm{B}$,
$\mathrm{CH} = \delta_3 = 2\mathrm{R}\cos\mathrm{C}$,
also $\mathrm{BD} = \epsilon_2$,
and $\mathrm{CD} = \epsilon_3$,
so that $\epsilon_2 + \epsilon_3 = a$;

then, (x, y) being the coordinates of the other focus referred to DC, DA, as axes, we have

$$\delta_2 \pm \surd\left\{(\epsilon_2 + x)^2 + y^2\right\} = \delta_3 \pm \surd\left\{(\epsilon_3 - x)^2 + y^2\right\},$$

whence
$$\pm\, 2\,(\delta_2 - \delta_3)\surd\left\{(\epsilon_2 + x)^2 + y^2\right\} = \epsilon_3{}^3 - \epsilon_2{}^2 - (\delta_3 - \delta_2)^2 - 2ax$$
$$= 2k_1{}^2 - 2ax, \text{ suppose};$$

$$\therefore\quad (\delta_3 - \delta_2)^2\, y^2 + \Big[(\delta_3 - \delta_2)^2 - a^2\Big]x^2 + 2x\Big[\,\epsilon_2\,(\delta_3 - \delta_2)^2 + k_1{}^2 a\Big]$$
$$+\,\epsilon_2{}^2\,(\delta_3 - \delta_2)^2 - k_1{}^4 = 0.$$

Shifting the axes to GC, GO, we have the equation

$$\frac{x^2}{(\delta_3 - \delta_2)^2} - \frac{y^2}{a^2 - (\delta_3 - \delta_2)^2} = \left\{\frac{a\epsilon_2 + k_1{}^2}{a^2 - (\delta_3 - \delta_2)^2}\right\}^2$$

which reduces to $\dfrac{x^2}{\sin^2\frac{1}{2}(\mathrm{B} - \mathrm{C})} - \dfrac{y^2}{\cos\mathrm{B}\cos\mathrm{C}} = 4\mathrm{R}^2\cos^2\frac{1}{2}\mathrm{A}$ ....(i.);

since $\delta_3 - \delta_2 = 4\mathrm{R}\cos\frac{1}{2}\sin\frac{1}{2}(\mathrm{B} - \mathrm{C})$, $a^2 - (\delta_3 - \delta_2)^2 = 16\mathrm{R}^2\cos^2\frac{1}{2}\mathrm{A}\cos\mathrm{B}\cos\mathrm{C}$,

$a\epsilon_2 + k_1{}^2 = a\epsilon_2 + 8\mathrm{R}^2\cos\frac{1}{2}\mathrm{A}\sin\frac{1}{2}(\mathrm{B} - \mathrm{C})\cos\mathrm{B} = 8\mathrm{R}^2\cos^2\frac{1}{2}\mathrm{A}\cos\mathrm{B}\cos\mathrm{C}$,

and $2\mathrm{GD} = \epsilon_3 - \epsilon_2 = 2\mathrm{R}\sin(\mathrm{B} - \mathrm{C})$.

From (i.) we see that, if the triangle is acute angled, the locus is a hyperbola for each of the three sides, but if one angle be obtuse then we have a hyperbola and two ellipses.

Equation (i.) is also equivalent to

$$\frac{4x^2}{(\delta_3 - \delta_2)^2} - \frac{y^2}{\delta_2\delta_3\cos^2\frac{1}{2}\mathrm{A}} = 1 \text{......................(ii.)};$$

hence if the semi-axes are a_1, b_1 we have $2a_1 = \delta_1 - \delta_2$, $b_1 = (\delta_3\delta_2)^{\frac{1}{2}}\cos\frac{1}{2}\mathrm{A}$.

If we take HS (perpendicular to GO) equal to 2DG, we get S a point on the hyperbola; also producing HG to meet the circumscribing circle in H′, we get H′ another point on the curve. Now, though the hyperbola is the locus of the foci, the converse does not hold good; thus, for ellipses, the branch to the right only satisfies the conditions. The other loci are readily constructed and relations between the eccentricities, &c., are readily obtainable.

The angular points of the triangle are the foci of the conic loci.

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WITH THEIR

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⁂ *Vols. I., II., III., IV., V., VI., VII. may still be had,*
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CORRIGENDA.

Vol. VII.

p. 69, end of line 12. *omit* "the".
p. 71, lines 3, 4, 5, 6, 12, 13 from bottom, *for* (1) *read* (−1).
p. 89, last line. *for* $\Delta =$ &c. *read* $\Delta' =$ &c.

Vol. VIII.

p. 22, line 26, *for* τ *read* π.
p. 35, line 2 from bottom, *insert* "point" *before* (ξ^{-1}) &c.
p. 46, line 3 from bottom, *for* $2dd$ *read* $2dd'$.
p. 88, line 3 from bottom, *for* 46 *read* 4.6, and *for* (3) *read* (2).
p. 100, line 13. *for* k *read* k^2.
p. 109, last line, *for* Hymer's *read* Hymers'.
p. 110, line 4 from top, *for* DM. DM *read* DM, DM'.
p. 110, line 4 from bottom. *for* CF *read* MH.

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Solved Questions.

1496. Prove that a triangular pyramid whose vertices are A, B, C, D, and a parallelepiped formed from it as follows, have the same centre of gravity; viz., through any point in each of the opposite edges AB and CD draw straight lines parallel to the other edge, we thus get two parallel plane faces of the parallelepiped; two other parallel faces of it are similarly obtained from the opposite edges AC and BD; and the third pair of faces are obtained from the remaining two opposite edges AD and BC. 87

No. Page

1805. Prove that in general

$$\int_0^l dx \int_0^x dy \, \mathrm{F}(x, y) = \int_0^l dx \int_0^x dy \, \mathrm{F}(l-y, l-x) \text{ ....(1)},$$

and hence show instantaneously that

$$\int_0^\pi d\theta \int_0^\theta d\theta' \, (\sin\theta \sin\theta')^{2i-1} \sin(\theta-\theta') = \frac{1 \,.\, 3 \,.\, 5 \ldots (4i-1)}{2 \,.\, 4 \,.\, 6 \ldots 4i} \frac{\pi}{i} \ldots (2);$$

and (3) apply the latter integral to prove that the chance of four points taken at random in a circle forming the apices of a reentrant quadrilateral is $\frac{35}{12\pi^2}$. 88

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2. Four points being taken at random within a sphere, find the chance that the sphere drawn through them will lie wholly within the given sphere.. 90

1849. Two points are taken at random, one on each side of a given diameter of a circle: what is the probability that the chord drawn through them shall not exceed a given length? 92

1910. AB is a given straight line upon which four points are taken at random. Find the chance that their anharmonic ratio (estimated by the quotient of the whole into the middle by the product of the extreme segments) shall exceed a given quantity. 43

1911. Given four points, and also the "conic of centres"—viz., the conic which is the locus of the centres of the several conics which pass through the four given points; then if a conic through the four given points has for its centre a given point on the conic of centres, it is required to find a construction for the asymptotes of this conic. 70

1917. Supposing the surface of a sphere to be made up of an indefinite number of points, and straight lines to be drawn through every two of those points, determine the law of density of this mass of lines as depending on the distance from the centre of the sphere. 41

1927. To find the conic of least eccentricity that can be drawn through four given points. 107

1947. Prove in any way that

$$41^{41} \equiv 1 \text{ (mod. 83), and } 82^{41} \equiv 82 \text{ (mod. 83).}$$

N.B.—Mod. in short for *modulus*, the first expression meaning that $41^{41}-1$ is exactly divisible by 83. 29

1959. If the roots of the equation $(a, b, c, d, e)(x, 1)^4 = 0$ are $\alpha, \beta, \gamma, \delta$, find the equation whose roots are

$(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)$, &c. 48

1971. In a given tetrahedron to find a set of rectangular coordinate axes, such that each axis shall pass through a pair of opposite edges...... 85

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1973. Draw a chord of one of two given circles so as to touch and be cut in a given ratio at its point of contact with the other.... 110

1985. Find what integral value of x renders $\dfrac{3x^7-29}{547}$ = a whole number, or else prove it impossible. 99

2005. Find the whole number x so that $3x^7-29$ may be divisible by 557 28

2117. A point P is given (1) on the base of a triangle, (2) on a circle; draw a transversal through another given point Q to cut (1) or (2) so that the intercept may subtend a given or a maximum angle at P. 96

2227. Form 2^n-1 symbols into triads, so that every pair of the symbols shall appear once in the triads. 32

2234. Find a curve such that the area $\int ydx$ bounded by two ordinates, whose mutual distance is constant, varies directly as the ordinate midway between them 24

2237. Prove that the equation of the osculating circle at the extremity of the major axis of the ellipse $r+r'=2a$ is

$$\frac{r^2}{a+c}+\frac{r'^2}{a-c}=2a. \qquad\qquad 67$$

2243. Place the numbers 1 to 36 in six rows, six in a row, so that each row, vertical, lateral, and diagonal, shall make 111. ... 68

2249. Prove that the sum of the ternary products of the six quantities

$$\left.\begin{array}{l}(a-b)^2(b-c)^2(c-d)^2(d-e)^2(e-a)^2\\(a-b)^2(b-d)^2(d-e)^2(e-c)^2(c-a)^2\\(a-b)^2(b-e)^2(e-c)^2(c-d)^2(d-a)^2\end{array}\right|\begin{array}{l}(a-d)^2(d-b)^2(b-c)^2(c-e^2)(e-a^2)\\(a-c)^2(c-d)^2(d-b)^2(b-e)^2(e-a)^2\\(a-c)^2(c-b)^2(b-e)^2(e-d)^2(d-a)^2\end{array}$$

is equal to the sum of the ternary products of the complementary six, viz.,

$$\left.\begin{array}{l}(a-c)^2(c-e)^2(e-b)^2(b-d)^2(d-a)^2\\(a-d)^2(d-e)^2(e-b)^2(b-c)^2(c-a)^2\\(a-e)^2(e-b)^2(b-c)^2(c-d)^2(d-a)^2\end{array}\right|\begin{array}{l}(a-b)^2(b-e)^2(e-d)^2(d-c)^2(c-a)^2\\(a-d)^2(d-e)^2(e-c)^2(c-b)^2(b-a)^2\\(a-b)^2(b-d)^2(d-c)^2(c-e)^2(e-a)^2\end{array}$$

.... 18

2252. Find the forms of ϕ and ψ which make the mean value of $\phi(x)$ from $x=a-c$ to $x=a+c$ take the form $\phi(a).\psi(c)$.... 24

2261. Sixty-three school-boys walk out three abreast for thirty-one days in succession; it is required to arrange them daily so that no two of them shall walk twice abreast. 76

2280. A Cartesian oval or ellipse is cut by an *axial* circle [*i.e.* a circle bisected by the axis] in two points whose vectorial coordinates referred to two of the foci are (r, r') and (s, s'). Show that if a circle concentric to the former be drawn *touching* the curve, the vectorial coordinates of either point of contact are $\left\{\frac{1}{2}(r+s), \frac{1}{2}(r'+s')\right\}$. 66

2310. 1. Prove that if ABCD be a quadrilateral circumscribed about a circle, and P, Q, R, S the points of contact, then AC, BD, RP, SQ pass through one point.

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2. Hence show that the locus of the intersection of the lines joining the ends of chords in a circle which pass through a given point is the polar of that point. 33

2321. Given a conic, to find four points such that all the conics through the four points may have their centres in the given conic 26

2324. If four roots of a quintic be in harmonic progression, prove that the following relation between the invariants holds, viz., $J^3 - 2^7 . 3^2 JK + 2^{13} . 3^3 L \equiv 0$, (where J, K, L are the fundamental invariants as given in Salmon's *Higher Algebra*, 2nd edition, pp. 184—187.) 29

2330. Describe a circle whose centre shall be on a given diameter of a given circle, to cut that circle orthogonally and to touch another given circle. 31

2336. Find the locus of points such that the chord of contact of tangents drawn to a given conic from any one of the points is of constant length. 50

2339. 1. If two triangles in a plane are similar, but right and left in position; then the lines drawn from the angular points of one triangle parallel, perpendicular, or isoclinal to the corresponding sides of the other triangle, meet in a point. Find its locus, if the first triangle remain fixed, while the second moves in any manner in the plane.

2. Analytically, we may assume the sums of the corresponding angles to vanish. 99

2340. The circles of curvature at the extremities of a pair of semiconjugate diameters of an ellipse are drawn; find (1) the envelop of their radical axis, and (2) the curve upon which their points of intersection lie.......................... 94

2343. A is any point within or without a conic, B any point on its polar, CD a fixed straight line. Tangents BC, BD are drawn cutting CD in C, D. AD, AC meet BC, BD in E, F; show that EF is a fixed straight line and meets CD on the polar of A. 63

2346. Through the angular points of the triangle of reference ABC, draw B′C′, C′A′, A′B′ respectively perpendicular to BC, CA, AB, and find the trilinear equation of the circle drawn round the triangle A′B′C′ 27

2356. A tetrahedron is thrown into the air and a small shot fired through it; show that the chance of one of the faces (which is marked) being perforated, is $2F \div S$, where F is the area of the marked face and S the whole surface of the tetrahedron. 90

2359. A point is taken at random in each side of a triangle ABC, so forming a second triangle DEF; determine the locus of those points within ABC which have a given probability p of being enclosed within the triangle DEF.......................... 53

2368. Show that the four straight lines represented by the equations $u=0$, $v=0$, $lu^2+2muv+nv^2=0$, will form a harmonic pencil if $8m^2=9ln$. 26

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2370. If three points be taken at random on the surface of a sphere, find the chance (1) that some one of the angles of the triangle they determine will be greater than the sum of the other two, (2) that the area will exceed a great circle of the sphere, and (3) that the area will exceed a given area 21

2371. (4.) If P, Q be two points taken at random within the triangle ABC, what is the chance that the points A, B, P, Q may form a convex quadrangle ?.. 36, 74

2376. 1. If ABC is a self-conjugate triangle of a conic, to which DP, DQ are tangents at P and Q, then any conic through ABCD cuts PQ harmonically.

2. Hence, if DM, DM′ are the perpendiculars on the axes, DM, DM′ bisect the angles PMQ, PM′Q. 110

2378. Find the average (1) of the areas of all the circles which can be drawn within the circumference of a given circle, and (2) of the volumes of all the spheres which can be drawn within the surface of a given sphere; also find the probability (3) that any one of the circles taken at random from (1) will exceed a given circle, and (4) that any one of the spheres taken at random from (2) will exceed a given sphere. 73

2386. If the side AD of a square be divided harmonically in B and C, and squares be placed on AB, BC, CD, falling without the square on AD, prove that the circles round the four squares pass through a common point. 95

2388. Si l'on désigne par a, b, c, a', b', c', les côtés d'un hexagone inscrit dans un cercle; par A la diagonale qui n'a aucune extrémité commune avec les côtés opposés a et a'; par B, C les deux autres diagonales analogues; on aura les deux équations suivantes :—

$$B^2Ca + B^2b'c - Ba'\,(a^2 - b'^2 - c'^2) + a'^2b'c' = C^2Ba' + C^2b'c' - Ca\,(a'^2 - b'^2 - c'^2) + a^2b'c',$$

$$B^2Ca' + B^2bc - Ba\,(a'^2 - b^2 - c^2) + a^2bc = C^2Ba + C^2bc - Cx'\,(a^2 - b^2 - c^2) + a'^2bc,$$

qui permettent d'établir une équation entre les côtés de l'hexagone et une des grandes diagonales. (J'appelle grande diagonale toute diagonale qui joint deux sommets opposés.)... 25

2395. Proofs are required of the following generalization of a well-known theorem:—If from any point on a conic lines be drawn to meet the three sides of any inscribed triangle, parallel to the diameters bisecting those sides respectively, the three points of intersection will be in the same straight line......... 44

2397. If a body of homogeneous fluid be thoroughly shaken and allowed to come to rest again, the chance that no particle of the fluid now occupies its original position is ϵ^{-1}, where ϵ is the base of Napierian logarithms. 55

2405. If from the middles of the arcs which subtend the three sides of an inscribed triangle, three sets of perpendiculars be drawn to the three sides of the triangle, and their feet joined, a new triangle will be formed; prove that the sum of the segments of the perpendiculars of the original triangle, between their

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point of intersection and the sides of the new triangle, is equal to the excess of the diameter of the circumscribing circle above the radius of the inscribed circle. 62

2406. Show that $\frac{\{(\Gamma n+1)\}^2}{\Gamma(2n+2)}$

$$(1)\ldots=2\left\{\frac{B_{n+1}}{n+1}-\frac{n}{1}\cdot\frac{B_{n+2}}{n+2}+\frac{n(n-1)}{1.2}\cdot\frac{B_{n+3}}{n+3}-\&c.\right\}$$

$$(2)\ldots=4\left\{(2^{n+2}-1)\frac{B_{n+2}}{(n+1)(n+2)}-\frac{n}{1}(2^{n+3}-1)\frac{B_{n+3}}{(n+2)(n+3)}+\&c.\right\},$$

where B is the Representative of Bernoulli's numbers......... 20

2416. Find the point in a plane the sum of the squares of whose distances from any number of given lines in the plane shall be the minimum. 75

2417. Denoting the roots of the cubic equation $x^3+px+q=0$ by α, β, γ, it is required to form the equation the roots of which are $\alpha+\beta\gamma, \beta+\gamma\alpha, \gamma+\alpha\beta$ 23

2418. If the vertices of a pentagon be A, B, C, D, E; and if a, b, c, d, e be the areas of the triangles EAB, ABC, BCD, CDE, DEA; show that the area P of the pentagon is given by the equation
$P^2-(a+b+c+d)P+(ab+bc+cd+de+ea)=0$. 109

2426. If F_n denote the fraction

$\frac{1+a^2+a^4\ldots+a^{2n}}{a+a^3+\ldots+a^{2n-1}}$, prove that, in general, if n is not $=1$,

$$(1)\ldots F_n>\frac{n+1}{n}+\left(\sqrt{a}-\frac{1}{\sqrt{a}}\right)^2,\ (2)\ldots F_n>\frac{1}{2}\cdot\frac{n+1}{n}\left(a+\frac{1}{a}\right),\ (3)\ldots\frac{n}{n+1}\cdot F_n$$

increases with n. 58

2429. An ellipse (A) and hyperbola (B) have the same transverse and conjugate diameters; prove that the polar of any point on (B) with reference to (A) touches (B). 97

2438. Two parabolas turned in opposite directions have the same axis and coincident vertices; if from the vertex as centre, with a diameter a mean proportional between the parameters of the curves, a circle be described, prove that the polar of any point on one with reference to the circle touches the other. 98

2444. Upon any three chords (AB, AC, AD) of a circle as diameters circles are drawn, cutting one another again in E, F, G; prove that EFG is a straight line.................................... 48

2447. If in the development of $\sin^{-1}.x$, viz.,

$$\frac{x^1}{1}+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1.3}{2.4}\cdot\frac{x^5}{5}\ \ldots\ldots\ldots$$

every number which occurs is increased by unity, prove that the result, *i.e.*,

$$\frac{x^2}{2}+\frac{2}{3}\cdot\frac{x^4}{4}+\frac{2.4}{3.5}\cdot\frac{x^6}{6}+\ldots\ldots\ldots$$

is equal to $\frac{1}{2}(\sin^{-1}x)^2$ 59

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2452. Prove that if P, Q be two points chosen at random inside a tetrahedron ABCD, the chance is an even one of ABCPQ forming the angles of a *convex* solid. 61

2459. If ω be a cube root of unity, express the roots of the equation

$$ax^3+3bx^2+\frac{3}{a}\left\{b^2+\omega(ac-b^2)\right\}x+\frac{1}{a^2}\left\{a^2d+3(\omega-1)(ab-b^3)\right\}=0,$$

in terms of α, β, γ, the roots of $ax^3+3bx^2+3cx+d=0$. 88

2460. Find the conic which is the locus of a point such that the lines joining it to a pair of conjugate foci are harmonicals to the conic. The circular points at infinity being considered as conjugate foci, the three conics thus obtained and the conic in question have four points in common. 62

2466. If four points A, B, C, D be either in the same plane or not, and if the three rectangles AB . CD, AC . DB, AD . BC be taken; the sum of any two of them is greater than the third, except when the points lie on the circumference of a circle. 86

2471. Let a given surface, having a convex boundary *of any form whatever*, be referred to its centre of gravity and the principal axes of rotation situated in its plane; and, corresponding to an abscissa x, let y, y' be the respective distances of the boundary above and below the axis; then if h, k denote the radii of gyration round the axes, and M the total area, we shall have

$$h^2=\int x^2dx\,\frac{y+y'}{M},\quad k^2=\tfrac{1}{3}\int dx\,\frac{y^3+y'^3}{M}.$$

Let also $$A=\int x^2dx\,\frac{y^3+y'^3}{M},\quad B=\int xdx\,\frac{y\int y^2dx+y'\int y'^2dx}{M}$$

and $$C=\tfrac{1}{3}A+3B\text{ ; then}$$

1. If *three* points be taken at random, on the given surface, the average area of the triangle connecting them, in parts of the total area, is $(\Delta)=\frac{h^2k^2+C}{M^2}$; and the average square of the area, expressed in parts of the square of the total area, is

$$(\Delta^2)=\frac{3}{2}\cdot\frac{h^2k_2}{M^2}.$$

2. If *four* points be taken at random, the probability that the quadrilateral shall be reentrant $=4(\Delta)$.

3. If *five* points be taken at random on the surface, the probability of

a convex pentagon $=1-10(\Delta)+10(\Delta^2)$,
one reentrant point $=10(\Delta)-20(\Delta^2)$,
two reentrant points $=10(\Delta^2)$. 100

2472. Through four points on a circle to draw a conic, such that an axis may pass through the centre of the circle. 105

2473. Show (1) how to plant 81 trees so as to form 800 rows of 3 trees in a row, and (2) how to plant 10 trees so as to form 10 rows of three trees in a row. 106

Unsolved Questions.

No.
2275. (Proposed by the Rev. J. BLISSARD.)—Required to show that if $f(x) = \frac{c_1 x}{1^2} - \frac{c_2 x(x-1)}{1.2^2} + \frac{c_3 x(x-1)(x-2)}{1.2.3^2} -$ &c., then $c_1, c_2, c_3 \ldots$ can be so determined that $\frac{1}{1} + \frac{1}{m+1} + \frac{1}{2m+1}$ &c. $+ \frac{1}{(n-1)m+1}$ may be equated to any one of the m functions, viz., $f(mn)$, $f(mn-1)$, $f(mn-2), \ldots f(mn-m+1)$, which functions therefore are all of equal value.

2276. (Proposed by T. COTTERILL, M.A.)—If tangents to a cubic at the points A B C D meet in a point T on the curve, a conic through T A B C will cut the cubic again in points collinear with D, the intersection of the tangents to the cubic at these points is on the conic, and is collinear with the intersection of the tangents to the conic at the same points and T.

2285. (Proposed by Messrs. HERMITE and SYLVESTER.)—(1.) Prove that the Jacobian of two binary quantics cannot have a less number of real linear factors than the difference between the number of such in the two quantics themselves separately taken.

(2.) Prove that if ϕ, ψ are three binary quantics, the number of real linear factors in F cannot exceed the number of such in

$$(\phi x + \psi y)\left(\phi \frac{d}{dy} - \psi \frac{d}{dx}\right) \mathrm{F}.$$

(3.) From (2) deduce (1), and obtain ROLLE's theorem as a particular case of the latter.

2289. (Proposed by C. TAYLOR, M.A.)—(1.) Interpret the tangential equation $pqr = 0$. (2.) Represent by a tangential equation a pair of straight lines regarded as a limit of a conic.

2301. (Proposed by W. K. CLIFFORD, B.A.)—A circle is drawn so that its radical axis with respect to the focus S of a parabola is a tangent to the parabola; if a tangent to the circle cut the parabola in A, B, and if SC, bisecting the angle ASB, cut AB in C, the locus of C is a straight line.

2312. (Proposed by Professor CAYLEY.)—*Trace* the curve defined by the equations given in the solution of Quest. 2110. (*Reprint*, Vol. VII., p. 17.)

2327. (Proposed by the late G. C. DE MORGAN, M.A.)—If $p_0, p_1, p_2 \ldots q_0, q_1, q_2 \ldots$ be whole numbers, such that $\frac{p_n}{q_n}$ has a limit finite or 0, the series $\frac{p_0}{q_0} - \frac{p_1}{q_0 q_1} + \frac{p_2}{q_0 q_1 q_2} - \ldots$ is an incommensurable quantity.

2334. (Proposed by W. S. BURNSIDE, M.A.)—1. Construct a triangle of given species, having its vertices on three concentric circles.

2. The triangle being equilateral, what relation exists between the radii when only one such triangle can be so constructed ?

No.

2337. (Proposed by Professor SYLVESTER.)—By the form probability of a group, understand the probability of a group of a given number of points satisfying a prescribed condition of *form*.

Let ABCD be any quadrilateral; ϕ the form probability of a group of points limited to the area ABCD; p, q, r, s the corresponding probabilities when one of the group is restricted to the lines AB, BC, CD, DA respectively. Show that if four forces act along the successive sides of ABCD, regarded as a rigid figure, with intensities proportional to $\phi - p$, $\phi - q$, $\phi - r$, $\phi - s$ respectively, they will keep the quadrilateral at rest.

2342. (Proposed by the Rev. R. TOWNSEND, F.R.S.)—Construct the minimum triangle of given species whose three sides shall touch three given circles.

2371. (Proposed by Professor CAYLEY.)—Consider a triangle ABC, and a point P; and let AP meet BC in M, and BP meet AC in N (if, to fix the ideas, P is within ABC, then M, N are in the sides BC, AC respectively, and the triangles APN, BPM are regarded as positive).

1. Find the locus of the point P, such that the ratio

(Δ APN + Δ BPM) : Δ ABC may have a given value.

2. Drawing from each point P, at right angles to the plane of the triangle, an ordinate PQ of a length proportional to the foregoing ratio (Δ APN + Δ BPM) : Δ ABC, trace the surface which is the locus of the point Q,—a surface which has, it is clear, the loci in (1) for its contour lines; and

3. Find the volume of the portion standing on ABC as base.

4. Deduce the solution of the following case of the four-point problem; viz., taking the points P, Q at random within the triangle ABC, what is the chance that the points A, B, P, Q may form a convex quadrangle? [For solutions of part (4) see *Reprint*, Vol. VIII., pp. 36, 74.]

2375. (Proposed by the Rev. W. ROBERTS, M.A.)—Being given a system of confocal spherical conics, in which the distance between the foci is $\frac{1}{2}\pi$, prove that the family of curves cutting them under an angle of 45° is algebraic, and determine its equation.

2381. (Proposed by W. S. BURNSIDE, M.A.)—1. Determine the degree of the locus represented by the equation

$$\frac{\alpha^2}{p^2} + \frac{\beta^2}{p_1^2} + \frac{\gamma^2}{p_2^2} - 2\frac{\beta\gamma}{p_1 p_2}\cos A - 2\frac{\gamma\alpha}{p_2 p}\cos B - 2\frac{\alpha\beta}{p p_1}\cos C = 1,$$

where α, β, γ are the distances of any point from the vertices of the triangle ABC, and p, p_1, p_2 the perpendiculars of the same triangle.

2. Show that the curve given by the above equation is the same as that referred to by Prof. CAYLEY in Quest. 2110. (*Reprint*, Vol. VII., p. 17.)

2389. (Proposed by Professor CREMONA.)—Deux droites qui divisent harmoniquement les trois diagonales d'un quadrilatère rencontrent en quatre points harmoniques toute conique inscrite dans le quadrilatère.

2390. (Proposed by the late G. C. DE MORGAN, M.A.)—

Prove that $\int_{-\infty}^{+\infty} \frac{1}{x}\,\phi\left(x - \frac{a}{x}\right)dx = 0$, a being anything positive; and

No.

$\int_{-\infty}^{+\infty}\frac{1}{x}\phi\left(x+\frac{a}{x}\right)dx=2\int_{-\infty}^{+\infty}\frac{\phi x}{x}dx$, a being infinitely small and positive, and ϕ being such a function that the subject of integration is finite for all finite values of x, however small a may be. In the second case, if $\frac{\phi x}{x}$ be infinite when $x=0$, the integral on the right must be replaced by $2\int_{-\infty}^{+\infty}\frac{1}{x}\left\{\phi x-\phi(-x)\right\}dx$.

2391. (Proposed by Professor SYLVESTER.)—Let μ points be given on a cubic curve. Through them draw any curve (simple or compound) of degree ν; the remaining $3\nu-\mu$ (say μ') points may be termed a first residuum to the given ones. Through these μ' points draw any curve of degree ν'; the remaining $3\nu'-\mu'$ points may be termed a residuum of the second order to the given ones; and in this way we may form at pleasure a series of residua of the third, fourth, and of any higher order. If μ is of the form $3i-1$, a residuum of the first or any odd order; and if μ is of the form $3i+1$, a residuum of the second or any even order in such series, may be made to consist of a single point, which I call the *residual* of the original μ points. Prove that any such residual is dependent wholly and solely on the original μ points, being independent of the number, degrees, and forms of the successive auxiliary curves employed to arrive at it.

2393. (Proposed by Dr. SALMON, F.R.S.)—1. What is the degree of the evolute of a circular cubic or bi-circular quartic? 2. Generally, how are the characteristics of the evolute affected when the curve is circular or bi-circular?

2399. (Proposed by the Rev. R. TOWNSEND, F.R.S.)—Find a point on a sphere such that the triangle determined by the middle points of the three arcs connecting it with three given points on the sphere shall have two of its sides given.

2403. (Proposed by W. S. BURNSIDE, M.A.)—Prove that confocal circular cubics cut at right angles: taking advantage of Prof. SYLVESTER's generation of these curves given in Quest. 1990. (*Reprint*, Vol. VI., p. 88.)

2408. (Proposed by S. ROBERTS, M.A.)—Representing by $R(x, y, z)$ the reciprocal of u, explain (1) the nature of the locus $R\left(\frac{du}{dx}, \frac{du}{dy}, \frac{du}{dz}\right)=0$, and account for its degree; and (2) give corresponding explanations relative to space of three dimensions.

2412. (Proposed by Professor CREMONA.)—La surface de STEINER ne contient pas de courbes d'ordre impair (ni réelles ni imaginaires); et toute courbe d'ordre $2n$ placée sur cette surface *peut être transformée rationnellement en* (*è punteggiata projettivamente a*) une courbe plane d'ordre n.

2413. (Proposed by Professor MANNHEIM.)—Three planes at right angles to each other, and intersecting in a point A, touch an ellipsoid whose centre is B. Prove that the connector AB and the normals at the three points of contact are generators of a hyperboloid.

No.

2421. (Proposed by M. W. CROFTON, B.A.)—Two places are taken at random in the northern hemisphere; find the chance of their distance exceeding 90° of a great circle.

2428. (Proposed by the Rev. J. BLISSARD.)—Prove that

$$(1)\ldots\frac{\Gamma(mx)}{\Gamma(nx)}\,(x=0)=\frac{n}{m},$$

$$(2)\ldots\frac{\Gamma(x)}{\Gamma(2x)}-\frac{\Gamma(x+2)}{\Gamma(2x+2)}\cdot\frac{\pi^2}{1.2}+\frac{\Gamma(x+4)}{\Gamma(2x+4)}\cdot\frac{\pi^4}{1.2.3.4}-\&c.=0.$$

2432. (Proposed by Captain A. R. CLARKE, R.E., F.R.S.)—Two perfectly flexible and inextensible films, being equal circles in form, are fastened together along the perimeter: what is the greatest amount of air that this could be made to hold (as an *air-cushion*)?

2458. (Proposed by S. WATSON.)—Three points A, B, C are taken at random on the circumference of a given circle; find the chance that a given point P within the circle lies within the triangle ABC.

2462. (Proposed by J. GRIFFITHS, M.A.)—Trace the curve represented by the tangential equation

$$(\sin A\,.\,\mu\nu+\sin B\,.\,\nu\lambda+\sin C\,.\,\lambda\mu)^2=4\lambda\mu\nu\,(\lambda\cos A+\mu\cos B=\nu\cos C).$$

2474. (Proposed by Professor EVERETT).—

If $\left(\frac{d}{dx}\right)^2+\left(\frac{d}{dy}\right)^2=0$, show that when n is any positive integer

$$\left(\cos\theta\,.\frac{d}{dx}+\sin\theta\,.\frac{d}{dy}\right)^n=\left(\cos n\theta\,.\frac{d}{dx}+\sin n\theta\,.\frac{d}{dy}\right)\left(\frac{d}{dx}\right)^n;\quad\text{and}$$

hence prove the theorem (due to Rankine and Stokes) that at every multiple point in a plane curve which fulfils the condition $\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}=0$, the branches made equal angles with each other.

2476. (Proposed by EXHUMATUS.)—A point is taken at random in each face of a tetrahedron ABCD; these points form an inscribed tetrahedron A′B′C′D′. Determine the locus of all those points within ABCD which have a given probability p of being within A′B′C′D′.

2482. (Proposed by H. R. GREER, B.A.)—A quadrilateral circumscribes an ellipse, the eccentric angles of the points of contact being α, β, γ, δ. Prove that the condition that a pair of opposite vertices should lie on a confocal conic may be thus expressed:

$$\begin{aligned}&a^2(\cos A+\cos B+\cos C+\cos D)(\cos A+\cos B-\cos C-\cos D)\times\\&\qquad(\cos C+\cos A-\cos B-\cos D)(\cos B+\cos C-\cos A-\cos D)=\\&b^2(\sin A+\sin B+\sin C+\sin D)(\sin A+\sin B-\sin C-\sin D)\times\\&\qquad(\sin C+\sin A-\sin B-\sin D)(\sin B+\sin C-\sin A-\sin D);\end{aligned}$$

where $A=\frac{1}{4}(-\alpha+\beta+\gamma+\delta)$, $B=\frac{1}{4}(\alpha-\beta+\gamma+\delta)$,
$C=\frac{1}{4}(\alpha+\beta-\gamma+\delta)$, $D=\frac{1}{4}(\alpha+\beta+\gamma-\delta)$.

Hence deduce a solution of Mr. C. Taylor's unsolved Question 1819 (*Reprint*, Vol. V., p. 16).

2492. (Proposed by Professor SYLVESTER.)—Prove that the mean distance between any two points taken arbitrarily within a triangle is a constant fraction of the mean of the mean distances of the angles from

No.

the several points in the opposite sides, and determine the value of the fraction. Show also that the same is true of the mean distance between two points in any parallelogram, and determine the value of the fraction for that case.

2495. (Proposed by W. S. B. Woolhouse, F.R.A.S.)—Equal segments are cut off from two opposite corners of a given rectangle, the lines of section being parallel to the diagonal; determine their magnitude such that if five points be taken at random on the remaining surface, the probability of their forming the apices of a convex pentagon shall be a maximum.

2502. (Proposed by J. Griffiths, M.A.)—If A, B, C, D be functions of x, y, z; and A_1, B_1, C_1, D_1 be certain constants; required the envelope of $A+\lambda B+\mu C+\nu D=0$, when the variables λ, μ, ν are connected by the two equations $A_1+\lambda B_1+\mu C_1+\nu D_1=0$, $\lambda^2+\mu^2+\nu^2-2\lambda\mu\nu=1$.

2503. (Proposed by H. R. Greer, B.A.)—Prove that the number of normals that can be drawn from a point in space of p dimensions to a (hyper)$^{p-3}$ surface (i.e., what corresponds to a *surface* in space of three dimensions) of the nth degree is $\frac{n}{n-2}\left\{(n-1)^p-1\right\}$.

2512. (Proposed by Professor Cayley.)—If B, C, Q, R are any given points, find the locus of a point A such that
AQ . AR . BC + BQ . BR . CA + CQ . CR . AB = BC . CA . AB.

N.B.—The construction of one point on the locus is given by the theorem, Quest. 2010 (*Reprint*, Vol. VII., p. 43).

2515. (Proposed by F. C. Mathews, M.A.)—A man has a weight of P lbs. which is broken into n pieces, each being integral in lbs.; find the least value of n such that it may be possible to weigh with the pieces all weights from 1 lb. up to P lbs. Find also the maximum value of P corresponding to any value of n; and find, furthermore, with any value of P and the corresponding minimum of n, the total number of distinct modes of partition which will satisfy the condition.

2520. (Proposed by Captain A. R. Clarke, F.R.S.)—Three points are taken at random in a given triangle; the probability (p) that they enclose a given point whose triangular coordinates are x, y, z, is

$$p = F(x, y, z) + F(y, z, x) + F(z, x, y),$$

where $F(x, y, z) = 2xyz\,(1-6x^2) + (6y^2z^2-24y^3z^3)\log\left(1+\frac{x}{yz}\right)$.

2523. (Proposed by T. Savage, M.A.)—1. If A, B, C be three closed areas situated in such a manner that no straight line can be drawn meeting more than two of them, the average area of all triangles formed by joining three points taken at random in A, B, C respectively is equal to the area of the triangle formed by joining the centres of gravity of A, B, C.

2. If A, B, C, D be four closed volumes such that (1) no straight line can meet more than two, and (2) no plane more than three, the average volume of the tetrahedron formed by joining four points taken at random in A, B, C, D respectively is equal to the volume of the tetrahedron formed by joining the centres of gravity of A, B, C, D respectively.

MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

AT RANDOM: No. IV.

By W. S. B. Woolhouse, F.R.A.S.

The opening paragraph of Mr. Godfray's remarks, No. III., upon the phrase "at random," (*Reprint*, Vol. VII., p. 98), intimating the necessity of strict definitions of random quantities, cannot of course be objected to, since mathematical investigations of every kind ought invariably to proceed upon accurate definitions. But I would here repeat my first statement, that the phrase "at random" has in general only one absolute and unqualified meaning; that the assemblage of quantities *all equally likely*, to which the random one belongs, should be elicited from the true nature of the problem, and not engrafted upon it as an arbitrary hypothesis, however feasible; and that mathematicians perfectly agree as to the precise meaning of "a line drawn at random from a given point," and of "a point taken at random on a given area."

The example relating to the projectile "thrown at random from a point" is not a bad illustration. Looking at the simple phraseology of the question, I unhesitatingly approve of the solution of the minority of candidates who took a hemisphere with the point as centre, and assumed that the chance of the direction of projection meeting any portion of the surface would be proportional to the area of that portion. According to the other solution, it is assumed that all azimuths are equally probable and all altitudes equally probable, so that all directions in space are not equally probable, the probability increasing towards the zenith. The former solution agrees with the supposition that the gun is worked on a pivot so as to move freely in all directions alike; the latter supposes it to be worked separately in azimuth and altitude, and it derives its chief support from the circumstance that the assumption is consistent with certain mechanical conditions, though at variance with the strict wording of the question.

It does not appear that anything more can be said on the subject, without trespassing unnecessarily upon your valuable pages; but if Mr. Godfray should consider it to admit of any further elucidation, I should like to know whether he really supposes the material discrepancies between the results of the different investigations of the "four point problem" have anything whatever to do with any existing misconceptions as to the simple definition of a "random point," or of any want of accord as to what that definition should be. This indeed was the principal theme of Mr. Godfray's first paper "On some Problems in the Theory of Chances," and I am still compelled to withhold my assent from what he had assumed, especially in reference to the "four point problem."

2249. (Proposed by Professor SYLVESTER.)—Prove that the sum of the ternary products of the six quantities

$$\begin{array}{l|l}
(a-b)^2(b-c)^2(c-d)^2(d-e)^2(e-a)^2 & (a-d)^2(d-b)^2(b-c)^2(c-e)^2)(e-a)^2 \\
(a-b)^2(b-d)^2(d-e)^2(e-c)^2(c-a)^2 & (a-c)^2(c-d)^2(d-b)^2(b-e)^2(e-a)^2 \\
(a-b)^2(b-e)^2(e-c)^2(c-d)^2(d-a)^2 & (a-c)^2(c-b)^2(b-e)^2(e-d)^2(d-a)^2
\end{array}$$

is equal to the sum of the ternary products of the complementary six, viz.,

$$\begin{array}{l|l}
(a-c)^2(c-e)^2(e-b)^2(b-d)^2(d-a)^2 & (a-b)^2(b-e)^2(e-d)^2(d-c)^2(c-a)^2 \\
(a-d)^2(d-e)^2(e-b)^2(b-c)^2(c-a)^2 & (a-d)^2(d-e)^2(e-c)^2(c-b)^2(b-a)^2 \\
(a-e)^2(e-b)^2(b-c)^2(c-d)^2(d-a)^2 & (a-b)^2(b-d)^2(d-c)^2(c-e)^2(e-a)^2
\end{array}$$

I. *Solution by the* PROPOSER.

Call ρ_1, ρ_2 the two sums in question, and let $a = 0$; then ρ_1 becomes

$$\Sigma L^2 + \Sigma M^2 + \Sigma N^2,$$

where $L = b^3cde.(c-d)^2(c-e)^2(d-e)^2(b-c)(b-d)(b-e)$,

also $M = b^2c^2de.(d-e)^3(b-c)^2(b-d)(b-e)(c-d)(c-e)$,

and $N = b^2c^2d^2.(d-e)^2(c-e)^2(b-e)^2(b-c)(c-d)(d-b)$,

the summation referring to the interchanges of b, c, d, e with one another.

Thus the effect of an interchange between any two of the letters b, c, d, e is to leave ρ_1 unaltered, and consequently ρ_1 and ρ_2 are identical whenever any one of the quantities a, b, c, d, e vanishes.

But in general $(\rho_1 - \rho_2)^2$ is obviously from its form an invariant of the function (with unity for its leading term) whose roots are a, b, c, d, e, and this invariant, if not zero, from what has been shown above, contains as a factor the last coefficient of such function: but no coefficient of a quantic can ever be contained in any of its invariants; for if so, an arbitrary linear function of all the coefficients would enter into the same. Hence $\rho_1 = \rho_2$.

NOTE.—M. HERMITE appears to have been the first to notice this property of the ternary products of the separate groups, in his memoir on the equation of the 5th Degree in the *Comptes Rendus* for the present year.

The sum of such products (say R) will easily be seen to have the property of vanishing whenever $a = b$ and $c = d$; for in such case 4 out of the 6 terms in either group become zero. It is therefore a linear function of A (*i.e.* $2^{11}L - J^3$), and JD; L, J, D having the meanings ascribed to them in my *Trilogy*, in the *Phil. Trans.* for 1865. From M. HERMITE's calculations the actual value of R appears to be a numerical multiple of $A - \frac{3}{8}JD$, so that by virtue of the rule given in the trilogy ($-\frac{3}{8}$ being comprehended within the limits -2 and 1) R is a good and sufficient third criterion along with J and D the first and second criteria for determining the nature of the roots of a quintic.

II. *Solution by* W. S. Burnside, M.A.

1. Write $\alpha^{-1}, \beta^{-1}, \gamma^{-1}, \delta^{-1}$ for $a-e, b-e, c-e, d-e$ respectively, and multiply all the terms by $(\alpha\beta\gamma\delta)^4$; then all the denominators disappear, and the first six quantities are thus reduced to

$$\begin{array}{c|c} (\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\delta)^2 & (\alpha-\delta)^2(\delta-\beta)^2(\beta-\gamma)^2 \\ (\alpha-\beta)^2(\beta-\delta)^2(\gamma-\alpha)^2 & (\alpha-\gamma)^2(\gamma-\delta)^2(\delta-\beta)^2 \\ (\alpha-\beta)^2(\gamma-\delta)^2(\delta-\alpha)^2 & (\alpha-\gamma)^2(\gamma-\beta)^2(\delta-\alpha)^2 \end{array}$$

and the other six are the quantities complementary to these in

$$(\alpha-\beta)^2(\alpha-\gamma)^2(\alpha-\delta)^2(\beta-\gamma)^2(\beta-\delta)^2(\gamma-\delta)^2.$$

Again write l, m, n for $\alpha-\delta, \beta-\delta, \gamma-\delta$ respectively, and the first six quantities properly arranged become

$$(1)\ \ldots\ldots\ldots\ldots\ldots\ldots \left\{\begin{array}{c|c} l^2(n-l)^2\ (m-n)^2 & m^2n^2(n-l)^2 \\ m^2(l-m)^2\ (n-l)^2 & n^2l^2\ (l-m)^2 \\ n^2(l-m)^2(m-n)^2 & l^2m^2(m-n)^2 \end{array}\right.$$

and the other six are the complementary quantities in

$$l^2m^2n^2(m-n)^2\ (n-l)^2\ (l-m)^2,$$

which may be arranged in the two following ways:—

$$(2)\ \ldots\ldots\ldots\ldots\ldots \left\{\begin{array}{c|c} l^2(m-n)^2(l-m)^2 & m^2n^2(l-m)^2 \\ m^2(m-n)^2(n-l)^2 & n^2l^2(m-n)^2 \\ n^2(l-m)^2(n-l)^2 & l^2m^2(n-l)^2 \end{array}\right\}$$

$$(3)\ \ldots\ldots\ldots\ldots\ldots \left\{\begin{array}{c|c} m^2(n-l)^2(m-n)^2 & l^2m^2(n-l)^2 \\ n^2(l-m)^2(n-l)^2 & m^2n^2(l-m)^2 \\ l^2(l-m)^2(m-n)^2 & l^2n^2(m-n)^2 \end{array}\right.$$

Now if any three terms in (1) be combined, the resulting constituent will be obtained from *one or other* of the arrangements (2) and (3) by combining the terms occupying the same positions as those selected in (1); and thus we see that the sum of the ternary combinations of both sets of six quantities are identical term by term.

2. When the 20 ternary combinations expressed in terms of $\alpha, \beta, \gamma, \delta$ are arranged symmetrically, we have

$$\begin{array}{ll} \Sigma\left\{(\alpha-\gamma)^4(\alpha-\delta)^4(\beta-\gamma)^4(\beta-\delta)^4\right\}.(\gamma-\delta)^2 \ldots\ldots\ldots\ldots & \text{six terms} \\ +\Delta\Sigma(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \text{four terms} \\ +\Delta\Sigma(\alpha-\delta)^2(\beta-\delta)^2(\gamma-\delta)^2\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \text{four terms} \\ +\Delta\Sigma(\alpha-\beta)^2(\gamma-\delta)^4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \text{six terms} \end{array}$$

where $\Delta = (\alpha-\beta)^2(\alpha-\gamma)^2(\alpha-\delta)^2(\beta-\gamma)^2(\beta-\delta)^2(\gamma-\delta)^2$.

3. Since the function we have been discussing is an invariant of the quintic having a, b, c, d, e for its roots, and is of the 12th degree in the coefficients, and of the 30th in the roots, it must necessarily be expressible in terms of the three fundamental invariants of the quintic, and must therefore be of the form $p.\mathrm{A}^3+q.\mathrm{AB}+r.\mathrm{C}$, (Salmon's *Higher Algebra*, Art. 186); and p, q, r are such that this function vanishes for a pair of double roots, viz. $a = b$, and $c = d$.

2406. (Proposed by Rev. J. Blissard.)—Show that $\dfrac{\{\Gamma(n+1)\}^2}{\Gamma(2n+2)}$

$$(1)\ldots = 2\left\{\frac{B_{n+1}}{n+1} - \frac{n}{1}\cdot\frac{B_{n+2}}{n+2} + \frac{n(n-1)}{1.2}\cdot\frac{B_{n+3}}{n+3} - \&c.\right\}$$

$$(2)\ldots = 4\left\{(2^{n+2}-1)\frac{B_{n+2}}{(n+1)\,(n+2)} - \frac{n}{1}(2^{n+3}-1)\frac{B_{n+3}}{(n+2)\,(n+3)} + \&c.\right\},$$

where B is the Representative of Bernoulli's numbers.

Solution by the Proposer.

By use of what I have called *Representative Notation*, it is easy to arrive at the following general formula, viz.,

$$\frac{(1+x)^{n+1}}{n+1} - \frac{m}{1}\cdot\frac{(1+x)^{n+2}}{n+2} + \frac{m(m-1)}{1.2}\cdot\frac{(1+x)^{n+3}}{n+3} - \&c.$$

$$+(-1)^{m+1}\left\{\frac{x^{m+1}}{m+1} + \frac{n}{1}\cdot\frac{x^{m+2}}{m+2} + \frac{n(n-1)}{1.2}\cdot\frac{x^{m+3}}{m+3} + \&c.\right\}$$

$$= \frac{\Gamma(m+1)\,\Gamma(n+1)}{\Gamma(m+n+2)}.$$

Let $m = n$, then $\dfrac{(1+x)^{n+1}}{n+1} - \dfrac{n}{1}\dfrac{(1+x)^{n+2}}{n+2} + \&c.\ldots$

$$+(-1)^{n+1}\left(\frac{x^{n+1}}{n+1} + \frac{n}{1}\cdot\frac{x^{n+2}}{n+2} + \&c.\right) = \frac{\{\Gamma(n+1)\}^2}{\Gamma(2n+2)}\ \ldots\ldots(1).$$

Now (1) for x put B, then since $(B+1)^r = B_r\ (r>1)$ and $=0$ (r odd), we have (See Theory of Generic Equations in the *Quarterly Journal of Mathematics*, Vol. IV.)

$$\frac{\{\Gamma(n+1)\}^2}{\Gamma(2n+2)} = \frac{B_{n+1}}{n+1} - \frac{n}{1}\frac{B_{n+2}}{n+2} + \frac{n(n-1)}{1.2}\frac{B_{n+3}}{n+3} - \&c.\ldots$$

$$+(-1)^{n+1}\left\{\frac{B_{n+1}}{n+1} + \frac{n}{1}\frac{B_{n+2}}{n+2} + \&c.\right\}$$

$$= 2\left\{\frac{B_{n+1}}{n+1} + \frac{n(n-1)}{1.2}\cdot\frac{B_{n+3}}{n+3} + \&c.\right\}\ n \text{ odd, and}$$

$$= -2\left\{\frac{n}{1}\frac{B_{n+2}}{n+2} + \frac{n(n-1)(n-2)}{1.2.3}\cdot\frac{B_{n+3}}{n+3} + \&c.\right\}\ n \text{ even.}$$

But B_{n+1}, B_{n+3}, B_{n+5} &c. all vanish (n even) and B_{n+2}, B_{n+4}, &c. all vanish n odd. Hence whether n be even or odd

$$\frac{\{\Gamma(n+1)\}^2}{\Gamma(2n+2)} = 2\left\{\frac{B_{n+1}}{n+1} - \frac{n}{1}\cdot\frac{B_{n+2}}{n+2} + \frac{n(n-1)}{1.2}\cdot\frac{B_{n+3}}{n+3} - \&c.\right\}$$

Ex. Let $n = 3$, then $\dfrac{\{\Gamma(4)\}^2}{\Gamma(8)}\left(=\dfrac{1}{140}\right) = 2\left(\dfrac{B_4}{4} - 3\dfrac{B_5}{5} + 3\dfrac{B_6}{6} - \dfrac{B_7}{7}\right);$

t $B_4 = -\frac{1}{30}$, $B_5 = 0 = B_7$, $B_6 = \frac{1}{42}$, $\therefore \frac{1}{140} = 2\left(-\frac{1}{120} + \frac{1}{84}\right)$, which holds.

Again (2), in (I) for x put A, where A belongs to the Generic Equation $\mathrm{A}+1)^r + \mathrm{A}^r = 0\,(r>0)$, and $\therefore (\mathrm{A}+1)^r = -\mathrm{A}_r$ which $= \frac{2^{r+2}-2}{r+1}\,\mathrm{B}_{r+1}$.

Hence

$$\frac{\{\Gamma(n+1)\}^2}{\Gamma(2n+2)} = (2^{n+3}-2)\frac{\mathrm{B}_{n+2}}{(n+1)(n+2)} - \frac{n}{1}(2^{n+4}-2)\frac{\mathrm{B}_{n+3}}{(n+2)(n+3)}$$

$$+ \frac{n(n+1)}{1.2}.(2^{n+5}-2)\frac{\mathrm{B}_{n+4}}{(n+3)(n+4)} - \&c.$$

$$+(-1)^{n+1}\left\{(2^{n+3}-2)\frac{\mathrm{B}_{n+2}}{(n+1)(n+2)} + \frac{n}{1}(2^{n+4}-2)\frac{\mathrm{B}_{n+3}}{(n+2)(n+3)} + \&c.\right\}$$

therefore, as in the preceding case, whether n be even or odd,

$$\frac{\{\Gamma(n+1)\}^2}{\Gamma(2n+2)} = 4\left\{(2^{n+2}-1)\frac{\mathrm{B}_{n+2}}{(n+1)(n+2)} - \frac{n}{1}(2^{n+3}-1)\frac{\mathrm{B}_{n+3}}{(n+2)(n+3)}\right.$$

$$\left. + \frac{n(n-1)}{1.2}(2^{n+4}-1)\frac{\mathrm{B}_{n+4}}{(n+3)(n+4)} - \&c.\right\}$$

Ex. (Let $n = 3$), then $\frac{1}{140} = 4\left(\frac{31\mathrm{B}_5}{4.5} - \frac{3.63\mathrm{B}_6}{5.6} + \frac{3.127\mathrm{B}_7}{6.7} + \frac{255\mathrm{B}_8}{7.8}\right)$

$$= 4\left(-\frac{3.63}{5.6}\cdot\frac{1}{42} + \frac{255}{7.8}\cdot\frac{1}{30}\right), \text{ which holds.}$$

2370. (Proposed by M. W. Crofton, B.A.)—If three points be taken at random on the surface of a sphere, find the chance (1) that some one of the angles of the triangle they determine will be greater than the sum of the other two, (2) that the area will exceed a great circle of the sphere, (3) that the area will exceed a given area.

Solution by Exhumatus.

1. Let A and B be two points taken at random on the surface of a sphere; then if the third point C be taken within the small circle which is described on AB as a diameter, it is clear (on joining C with the middle point of AB) that the angle at C is greater than the sum of the angles at A and B. Hence the chance that the angle C is greater than (A + B) is

$$p = \frac{1}{16\pi^2}\int_0^{\pi}(2\pi\sin c\,.\,dc)\;2\pi\left(1-\cos\frac{c}{2}\right)$$

$$= -\int_p^{\pi}\left(\cos\frac{c}{2} - \cos^2\frac{c}{2}\right)d\,.\cos\frac{c}{2} = \frac{1}{6}.$$

Consequently, *the chance that some one of the angles will exceed the sum of the other two is* $\frac{1}{4}$.

2. It is easily proved by the ordinary formulæ of Spherical Trigonometry that if the area of a spherical triangle on a given base AB be constant, the locus of the vertex is a small circle. If the given area be a great circle of the sphere, the locus is a small circle described on a diameter equal and opposite to AB. If a third point C be taken within this small circle, the area of the triangle ABC will exceed a great circle of the sphere; and the probability of this by the foregoing case is $\frac{1}{8}$, which is therefore *the chance that the area will exceed a great circle of the sphere.*

3. Generally, to find the chance that the area will be less than a given area σ; let ρ be the length of an arc of a great circle joining C to the middle of AB; then it may be shown that

$$\tan\frac{\sigma}{2} = \frac{\sin\frac{1}{2}c \sin\rho \sin\phi}{\cos\frac{1}{2}c + \cos\rho},$$

where ϕ is the angle ρ makes with AB at their intersection. This equation is that of a small circle of the sphere whose centre is in the great circle which bisects AB at right angles, and of which the radius is determined by the equation $\tan r = \tan\frac{1}{2}c \operatorname{cosec}\frac{1}{2}\sigma$. This small circle cuts the great circle AB in two points A′, B′ directly opposite to A and B. It is clear that the point C must not fall within the circumference of this small circle, nor within the circumference of *another* small circle equal to it and similarly situated on the other side of AB, cutting AB also in the same two points A′, B′. The total area then to which C is restricted is $4\int_0^{\frac{1}{2}\pi}(1-\cos\rho)\,d\phi$, which may be integrated by expressing $\cos\rho$ in terms of ϕ, from the above expression for $\tan\frac{1}{2}\sigma$. Or thus; if M be the middle point of AB, and P the pole of the small circle, the area of the spherical triangle MPB′ is $\frac{1}{2}\pi + (\tau - \zeta) + (\frac{1}{2}\pi + \frac{1}{2}\sigma) - \pi$, where ζ is half the angle A′PB′, and $\cos r \tan\zeta = \tan\frac{1}{2}\sigma$. That the angle A′B′P is equal to $\frac{1}{2}\pi - \frac{1}{2}\sigma$ is evident from the expression $\tan r = \tan\frac{1}{2}c \operatorname{cosec}\frac{1}{2}\sigma$. From the spherical triangle MB′P we have to deduct that part of the small circle which falls within it; the area of this is $(1-\cos r)(\pi - \zeta)$; the remainder multiplied by 4 is the area to which the third point C is restricted. Hence the chance required is

$$p = \frac{1}{\pi}\int_0^{\frac{1}{2}\pi}\left\{\frac{\sigma}{2} + \cos r\,(\pi - \zeta)\right\}\frac{\sin c}{2}\,dc.$$

Substitute here ζ for c by means of the equation $\cos\zeta = \cos\frac{1}{2}c\cos\frac{1}{2}\sigma$, and the expression is transformed to

$$p = \frac{2}{\pi\cos^2\frac{1}{2}\sigma}\int\left\{\frac{\sigma}{2}\cos\zeta\sin\zeta + (\pi - \zeta)\tan\frac{\sigma}{2}\cos^2\zeta\right\}d\zeta,$$

the limits being from $\zeta = \frac{1}{2}\sigma$, to $\frac{1}{2}\pi$. The result is

$$p = \frac{1}{\pi\cos^2\frac{1}{2}\sigma}\left\{\frac{\sigma}{2} + \frac{1}{2}\sin\frac{\sigma}{2}\cos\frac{\sigma}{2} - \pi\sin^2\frac{\sigma}{2} + \left(\frac{\pi}{2} - \frac{\sigma}{2}\right)\left(\frac{3}{4}\pi - \frac{\sigma}{4}\right)\tan\frac{\sigma}{2}\right\},$$

which is *the chance that the area will be less than* σ.

4. When the given area is that of a great circle of the sphere, we have $\sigma = \pi$ in (3), and p takes the apparent form of infinity, but is really a fraction of the form $\frac{0}{0}$ whose value is readily determined if we put $\sigma = \pi - 2\theta$.

For this value of σ we get

$$p = \frac{1}{\pi \sin^2 \theta}\left\{\pi \sin^2 \theta - \pi + \frac{\pi}{2} - \theta + \frac{1}{2} \sin \theta \cos \theta + \frac{\theta \cos \theta}{\sin \theta}\left(\frac{\pi}{2} + \frac{\theta}{2}\right)\right\}$$

$$= 1 - \frac{\sin \theta - \theta \cos \theta}{2 \sin^3 \theta} - \frac{\theta}{2\pi}\left\{\frac{\theta - \sin \theta \cos \theta}{\theta \sin^2 \theta} + \frac{\sin \theta - \theta \cos \theta}{\sin^3 \theta}\right\}.$$

Now, when θ is very small, it is easily seen that ultimately

$$\frac{\sin \theta - \theta \cos \theta}{\sin^3 \theta} = \frac{1}{3}, \text{ and } \frac{\theta - \sin \theta \cos \theta}{\theta \sin^2 \theta} = \frac{2}{3};$$

so that when $\theta = 0$, that is when $\sigma = \pi$, the required probability is $\frac{5}{6}$. *The chance that the area will exceed a great circle of the sphere is consequently* $\frac{1}{6}$.

2417. (Proposed by the Rev. W. ROBERTS, M.A.)—Denoting the roots of the cubic equation $x^3 + px + q = 0$ by α, β, γ, it is required to form the equation the roots of which are $\alpha\beta + \gamma$, $\beta\gamma + \alpha$, $\alpha\gamma + \beta$.

I. *Solution by the* REV. ROBERT HARLEY, F.R.S.

More generally, let α, β, γ be the roots of the complete cubic equation $x^3 + bx^2 + cx + d = 0$; then the equation $y^3 + By^2 + Cy + D = 0$, whose roots y_1, y_2, y_3, are $\alpha\beta + \gamma$, $\alpha\gamma + \beta$, $\beta\gamma + \alpha$, is readily calculated. For we have

$$\begin{aligned} -B &= \Sigma y = \Sigma\alpha\beta + \Sigma\alpha = c - b, \\ C &= \Sigma y_1 y_2 = \alpha\beta\gamma\Sigma\alpha + \Sigma\alpha^2\beta + \Sigma\alpha\beta \\ &= b(d-c) + 3d + c, \\ -D &= y_1 y_2 y_3 = \alpha^2\beta^2\gamma^2 + \alpha\beta\gamma\Sigma\alpha^2 + \Sigma\alpha^2\beta^2 + \alpha\beta\gamma \\ &= (d+c)^2 - bd(b+2) - d. \end{aligned}$$

Therefore the required equation is

$$y^3 - (c-b)y^2 + \left\{b(d-c) + 3d + c\right\} y - (d+c)^2 + bd(b+2) + d = 0.$$

In the particular case proposed, we have $b = 0$, and writing p and q in of c and d respectively, the equation becomes

$$y^3 - py^2 + (3q+p)y + q - (q+p)^2 = 0.$$

II. *Solution by* J. McDOWELL, M.A., F.R.A.S.; J. DALE; *and others.*

We have $$x^3 + px + q = 0 \quad\text{.........................}\quad (1);$$

and since $$\alpha\beta + \gamma = \frac{\alpha\beta\gamma}{\gamma} + \gamma = \frac{-q}{\gamma} + \gamma,$$

if we put $$y = \frac{-q}{x} + x, \text{ or } x^2 - yx - q = 0 \quad\text{...................}\quad (2),$$

and eliminate x between (1) and (2), the resulting equation in y will be the required one.

(1) $-x$ (2) gives $$yx^2 + (p+q)x + q = 0 \quad\text{..................}\quad (3);$$

$(3) - y\,(2)$ gives $(y^2 + p + q)\,x + q\,(y+1) = 0$, or $x = -\dfrac{q\,(y+1)}{y^2 + p + q}$.
This value of x substituted in (2) gives, after a few simple reductions,

$$y^3 - py^2 + (p + 3q)\,y + q - (p + q)^2 = 0,$$

the required equation.

2234. (Proposed by Professor EVERETT.) — Find a curve such that the area $\int y\,dx$ bounded by two ordinates, whose mutual distance is constant, varies directly as the ordinate midway between them.

Solution by THOMAS SAVAGE, M.A.

Let $y = f'(x)$ be the equation to the curve, $2a$ the given mutual distance of the two ordinates.

Then $$f(x + a) - f(x - a) = mf'(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

or $$2\left\{f'(x)\,a + f'''(x)\,\frac{a^3}{\Pi(3)} + f^{v}(x)\,\frac{a^5}{\Pi(5)} + \ldots\right\} = mf'x,$$

or $$\frac{f'''(x)}{f'(x)}\,\frac{a^3}{\Pi(3)} + \frac{f^{v}(x)}{f'(x)}\,\frac{a^5}{\Pi(5)} + \ldots = \frac{m}{2} - a.$$

If $\frac{1}{2}m = a$, this equation is satisfied by $f'''(x) = 0$, which gives us any straight line. If not, and k^2 be determined by either of the equations

$$\pm\, k^2\,\frac{a^3}{\Pi(3)} + k^4\,\frac{a^5}{\Pi(5)} \pm k^6\,\frac{a^7}{\Pi(7)} + \ldots = \frac{m}{2} - a$$

the equation (1) will be satisfied by $\dfrac{f'''(x)}{f'(x)} = \pm\, k^2$, which gives

$$y = \mathrm{A}\epsilon^{kx} + \mathrm{B}\epsilon^{-kx}, \text{ or } y = \mathrm{C}\sin(kx + \mathrm{D}),$$

where A, B, C, D are any arbitrary constants. The linear form of equation (1) shows that the sum or difference of any particular solution is also a solution.

2252. (Proposed by Professor EVERETT.)—Find the forms of ϕ and ψ which make the mean value of $\phi(x)$ from $x = a - c$ to $x = a + c$ take the form $\phi(a)\,.\,\psi(c)$.

Solution by the PROPOSER.

The mean value in question is $\dfrac{1}{2c}\displaystyle\int_{a-c}^{a+c}\phi(x)\,dx$, which, if F denote the primitive of ϕ, is equal to $\dfrac{1}{2c}\left\{\mathrm{F}(a + c) - \mathrm{F}(a - c)\right\}$; and this by Taylor's theorem is equal to the series $\phi(a) + \phi''(a)\,\dfrac{c^2}{\Pi(3)} + \phi'''(a\,\dfrac{c^4}{\Pi(5)} + \ldots\ldots$

Equating $\phi(a)\,.\,\psi(c)$ to this series, differentiating twice with respect to c and then putting $c=0$, we have $\phi(a)\,.\,\psi''(0) = \frac{1}{3}\phi''(a)$. Hence the ratio of $\phi''(a)$ to $\phi(a)$ is to be independent of the value of a. The only functions which fulfil this condition are

$\phi(x) = k\sin(mx+l)$ and $\phi(x) = k\epsilon^{mx} + l\epsilon^{-mx}$, k, l, m being any constants. Substituting these values in the series which is to be equal to $\phi(a)\,.\,\psi(c)$, we find that

when $\phi(x) = k\sin(mx+l)$, we have $\psi(c) = \dfrac{\sin mc}{mc}$;

when $\phi(x) = k\epsilon^{mx} + l\epsilon^{-mx}$, we have $\psi(c) = \dfrac{1}{2mc}\left(\epsilon^{mc} - \epsilon^{-mc}\right)$.

[The solution of Quest. 2234 follows at once from the above by putting $a-c$ and $a+c$ for the extreme values of x. The equation of the curve is $y = \phi(x)$, where $\phi(x)$ has either of the values found above.]

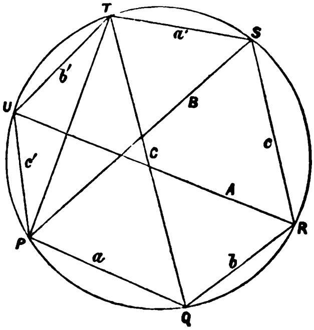
2388. (Proposed by E. Prouhet.)—Si l'on designe par a, b, c, a', b', c', les côtés d'un hexagone inscrit dans un cercle; par A la diagonale qui n'a aucune extrémité commune avec les côtés opposés a et a'; par B, C les deux autres diagonales analogues; on aura les deux équations suivantes:—

$$B^2Ca + B^2b'c - Ba'(a^2-b'^2-c'^2) + a'^2b'c' = C^2Ba' + C^2b'c' - Ca(a'^2-b'^2-c'^2) + a^2b'c',$$

$$B^2Ca' + B^2bc - Ba(a'^2-b^2-c^2) + a^2bc = C^2Ba + C^2bc - Ca'(a^2-b^2-c^2) + a'^2bc,$$

qui permettent d'établir une équation entre les côtés de l'hexagone et une des grandes diagonales. (J'appelle grande diagonale toute diagonale qui joint deux sommets opposés.)

Solution by S. Watson; G. A. Ogilvie; *and others.*



Let PQRSTU be the hexagon, and join PT; then $\angle PQT = PST = \pi - PUT = \alpha$ (suppose);

$$\therefore\ PT^2 = a^2 + C^2 - 2aC\cos\alpha = a'^2 + B^2 - 2a'B\cos\alpha = b'^2 + c'^2 + 2b'c'\cos\alpha;$$

therefore, by eliminating $\cos\alpha$,

$$(b'c' + aC)(b'^2 + c'^2 - a'^2 - B^2) = (b'c' + a'B)(b'^2 + c'^2 - a^2 - C^2) \ldots (1).$$

Similarly, by joining QS, we get

$$(bc + a'C)(b^2 + c^2 - a^2 - B^2) = (bc + aB)(b^2 + c^2 - a'^2 - C^2) \ldots (2).$$

Equations (1) and (2) reduce to the forms given in the question.

2321. (Proposed by Professor CAYLEY.)—Given a conic, to find four points such that all the conics through the four points may have their centres in the given conic.

Solution by SAMUEL ROBERTS, M.A.

If we take any three points forming a triangle of reference, and (l, m, n) are the coordinates of any other point, a quadrangle, of which the three points first taken are the intersections of opposite sides, is determined by

$$(l, m, n),\ (l, m, -n),\ (l, -m, n),\ (-l, m, n).$$

This convenient result gives for any conic circumscribing the quadrangle

$$A\alpha^2 + B\beta^2 + c\gamma^2 = 0,\quad Al^2 + Bm^2 + Cn^2 = 0.$$

Now the centre of this conic is given by

$$x : y : z = \frac{a}{A} : \frac{b}{B} : \frac{c}{C} \text{ or } \frac{a}{x} : \frac{b}{y} : \frac{c}{z} = A : B : C,$$

a, b, c representing the sides of the triangle of reference.

Hence the locus of centres is

$$\frac{al^2}{x} + \frac{bm^2}{y} + \frac{cn^2}{z} = 0.$$

If then we refer the given conic to any inscribed triangle and its equation is

$$\frac{A'}{x} + \frac{B'}{y} + \frac{C}{z} = 0,$$

four points fulfilling the condition imposed are determined by

$$\pm\sqrt{\left(\frac{A'}{a}\right)} : \pm\sqrt{\left(\frac{B'}{b}\right)} : \pm\sqrt{\left(\frac{C'}{c}\right)}.$$

2368. (Proposed by X. U. J.)—Show that the four straight lines represented by the equations $u=0$, $v=0$, $lu^2 + 2muv + nv^2 = 0$, will form a harmonic pencil if $8m^2 = 9ln$.

I. *Solution by* J. WALMSLEY.

Putting $k = \dfrac{2m}{3l}$ and $u' \equiv u + kv$, we have $2m = 3lk$ and $n = 2lk^2$; and the four straight lines are easily found to be, in another order,

$$u'=0,\quad u'-kv=0,\quad v=0,\quad u'+kv=0.$$

II. *Solution by* JAMES DALE.

The anharmonic ratio of the pencil formed by the four straight lines

$$u + k_1v = 0,\quad u + k_2v = 0,\quad u + k_3v = 0,\quad u + k_4v = 0$$

is

$$\frac{(k_1-k_2).(k_3-k_4)}{(k_1-k_4).(k_3-k_2)}.$$

Putting $k_1 = 0$ and $k_2 = \frac{1}{0}$, the anharmonic ratio of the pencil

$$u = 0,\ v = 0,\ u + k_3 v = 0,\ u + k_4 v = 0$$

is $\dfrac{k_3 - k_4}{-k_4}$, and if this be a harmonic ratio, then $k_3 = 2k_4$.

Now from the equation $lu^2 + 2muv + nv^2 = 0$, we get

$$k_3 = \frac{1}{l}\left\{m + \surd(m^2 - ln)\right\},\ k_4 = \frac{1}{l}\left\{m - \surd(m^2 - ln)\right\};$$

and substituting these values in $k_3 = 2k_4$, and freeing from radicals, we get

$$8m^2 = 9ln.$$

2346. (Proposed by the Rev. R. H. WRIGHT, M.A.)—Through the angular points of the triangle of reference ABC, draw B′C′, C′A′, A′B′ respectively perpendicular to BC, CA, AB, and find the trilinear equation of the circle drawn round the triangle A′B′C′.

I. *Solution by the* PROPOSER; E. MCCORMICK; *and others.*

The coordinates of the vertices of the triangle A′B′C′ are

$$\begin{array}{llll} A' \ldots\ldots & \alpha_1 = c \cot A \cos B, & \beta_1 = c \operatorname{cosec} A, & \gamma_1 = -c \cot A; \\ B' \ldots\ldots & \alpha_2 = -a \cot B, & \beta_2 = a \cot B \cos C, & \gamma_2 = a \operatorname{cosec} B; \\ C' \ldots\ldots & \alpha_3 = b \operatorname{cosec} C; & \beta_3 = -b \cot C; & \gamma_3 = \cot C \cos A. \end{array}$$

Now any circle may (Salmon's *Conics*, Art. 128) be represented by

$$(l\alpha + m\beta + n\gamma)(\alpha \sin A + \beta \sin B + \gamma \sin C) + \beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C = 0;$$

and substituting for (α, β, γ) their respective values at A′, B′, C′ we obtain three equations in terms of l, m, n from which we get

$$l = -\frac{\cos C}{\sin A \sin C},\ m = -\frac{\cos A}{\sin A \sin B},\ n = -\frac{\cos B}{\sin B \sin C};$$

and thus the required equation to the circle round A′B′C′ becomes

$$\alpha^2 \cot C + \beta^2 \cot A + \gamma^2 \cot B + \left(\cot A \frac{\sin C}{\sin B} + \sin A + \frac{\cos B}{\sin C}\right)\beta\gamma$$
$$+ \left(\cot B \frac{\sin A}{\sin C} + \sin B + \frac{\cos C}{\sin A}\right)\gamma\alpha + \left(\cot C \frac{\sin B}{\sin A} + \sin C + \frac{\cos A}{\sin B}\right)\alpha\beta = 0.$$

II. *Solution by* S. WATSON; *the* REV. J. L. KITCHIN, M.A.;
and many others.

The trilinear equations of B′C′, C′A′, A′B′, are

$$\alpha \cos C + \beta = 0,\ \beta \cos A + \gamma = 0,\ \gamma \cos B + \alpha = 0;$$

and the perpendiculars from any point $(\alpha', \beta', \gamma')$ on these lines, are respectively (Salmon's *Conics*, Art. 61),

$$x = \frac{\alpha' \cos C + \beta'}{\sin C}, \quad y = \frac{\beta' \cos A + \gamma'}{\sin A}, \quad z = \frac{\gamma' \cos \beta + \alpha'}{\sin B}.$$

Now, since the angles A′, B′, C′ are respectively equal to A, B, C, the trilinear equation of the circle about A′B′C′ in terms of (x, y, z) is

$$yz \sin A + zx \sin B + xy \sin C = 0;$$

hence, substituting and dropping the accents, the required equation is found to be

$$\frac{\operatorname{cosec} A}{\alpha + \gamma \cos B} + \frac{\operatorname{cosec} B}{\beta + \alpha \cos C} + \frac{\operatorname{cosec} C}{\gamma + \beta \cos A} = 0.$$

2005. (Proposed by MATTHEW COLLINS, B.A.)—Find the whole number x so that $3x^7 - 29$ may be divisible by 557.

I. *Solution by* SAMUEL BILLS.

I am not aware that this question admits of any *direct* solution. It is well known that if the expression $3x^7 - 29$ is divisible by 557 at all, it will be so for some value of x, positive or negative, not greater than $\frac{557-1}{2}$, or 278. The only method that I know of is to proceed by *trials*. Now by using various artifices and contractions I have tried *all* the numbers, both *positive* and *negative*, as far as 278; and I find that either $x = -35$, or $x = 522$ will answer the condition. If we take m = any whole number either positive or negative we shall have, generally, $x = 557m - 35$; from which any number of answers may be found.

I may observe that -35 and 522 are the *only* numbers *less* than 557 which will satisfy the condition. If $m = 2$, $x = 1079$; if $m = -1$, $x = -592$, &c. &c.

II. *Solution by* JAMES DALE.

If $3x^7 - 29 = M\,(557)$ be solvable in whole numbers, M must be a term of the series 2, 5, 8, ... $2 + (r-1)\,3$, and the equation may be reduced to the form

$$x^7 - 381 = N\,(557),$$

M and N being connected by the equation $M = 3N + 2$.

To solve the equation $x^7 - 381 = N\,(557)$, let $7p = q\,(557 - 1) + 1$; then the least values of p and q in positive integers are $p = 159$, $q = 2$.

Put $x = 381^{159}$, then $x^7 = 381^{1113}$, and

$$x^7 - 381 = 381\,(381^{1112} - 1) = 381\,(381^{556} + 1)\,.\,(381^{556} - 1).$$

Now by Fermat's Theorem $(381^{556} - 1) = (381^{557-1} - 1)$ is divisible by 557; therefore $x = 381^{159}$ is a root of the equation $x^7 - 381 = N\,(557)$, and hence also of the given equation $3x^7 - 29 = M\,(557)$.

Other values may be obtained by taking suitable values for p and q, such as $x = 381^{1:71}$, &c.

2324. (Proposed by W. S. BURNSIDE, M.A.)—If four roots of a quintic be in harmonic progression, prove that the following relation between the invariants holds, viz., $J^3-2^7.3^2\,JK+2^{13}.3^3\,L\equiv 0$, (where J, K, L are the fundamental invariants as given in Salmon's *Higher Algebra*, 2nd edition, pp. 184—187.)

Solution by the PROPOSER.

In this case the quintic may be reduced to the form

$$5\left\{lx^5+mx^4y+lxy^4+my^5\right\};$$

then $L\equiv(5^4+18.5^2-3^3)(lm)^6-4.5^3\,l^2m^2\,(l^8+m^8)$;
also $K\equiv(5^4+3^3)(lm)^4-2.5^3\,(l^8+m^8)$, and $J=2^7.3\,l^2m^2$.
Eliminating l and m we obtain the relation in the question.

1947. (Proposed by W. B. DAVIS, B.A.)—Prove in any way that

$$41^{41}\equiv 1 \text{ (mod. 83), and } 82^{41}\equiv 82 \text{ (mod. 83)}.$$

N.B.—Mod. is short for *modulus*, the first expression meaning that $41^{41}-1$ is exactly divisible by 83.

I. *Solution by* M. JENKINS, B.A.; W. H. LAVERTY; *and others.*

Since 41 is odd, $82^{41}+1$ is divisible by $82+1$,
therefore $82^{41}\equiv-1\equiv 82$ (mod. 83), which proves the 2nd congruence.
Again; $(2n)^n-(-1)^n$ is divisible by $2n-(-1)$, i.e. by $2n+1$ ...... (1);
and, by a well known theorem, if $2n+1$ be a prime number,

$$2^n\equiv 1\left\{\text{mod. }(2n+1)\right\}, \text{ if } 2n+1 \text{ be of the form } 8m\pm 1$$
$$\equiv -1 \text{ .... if .... } 8m\pm 3;$$

that is, $2^n-(-1)^{\frac{1}{2}n(n+1)}$ is divisible by $(2n+1)$ .... (2).
Multiplying (1) by $(-1)^n$, (2) by $(-1)^{\frac{1}{2}n(n+1)}$ and subtracting,

$$(-1)^n(2n)^n-(-1)^{\frac{1}{2}n(n+1)}\,2^n \text{ is divisible by } (2n+1);$$

and dividing by $(-1)^n\,2^n$, $n^n-(-1)^{\frac{1}{2}n(n-1)}$ is divisible by $(2n+1)$.
This shows that, if n be of the form $4m$ or $4m+1$,

$$n^n\equiv 1\,[\text{mod. }(2n+1)];$$

therefore, since 41 is of the latter form, and 83 a prime number,

$$41^{41}\equiv 1 \text{ (mod. 83)}.$$

II. *Solution by* J. DALE; S. W. BROMFIELD; *and others.*

1. By Fermat's Theorem, we have

$$41^{82}-1=(41^{41}+1)(41^{41}-1)=M\,(83),$$

or
$$\left\{(83-1)^{41}+2^{41}\right\}\left\{(83-1)^{41}-2^{41}\right\}=2^{41}.M\,(83).$$

Expanding $(83-1)^{41}$, every term except the last is a multiple of 83, and the above equation may be written

$$\{M(83)+(2.2^{40}-1)\}\{M(83)-(2.2^{40}+1)\} = 2^{41}.M.(83),$$

therefore either $(2.2^{40}-1)$ or $(2.2^{40}+1)$ is divisible by 83; but $(2.2^{40}-1)$ is not divisible by 83, (see Legendre, sect. 145); therefore $2.2^{40}+1$ is divisible by 83, and consequently $41^{41}-1$, and not $41^{41}+1$, is a multiple of 83.

2. We have

$$\begin{aligned} 82^{41}-82 &= 82(82^{40}-1) \\ &= 82(82^{20}+1)(82^{20}-1) \\ &= 82(82^{20}+1)(82^{10}+1)(82^{10}-1) \\ &= 82(82^{20}+1)(82^{10}+1)(82^{5}-1)(82^{5}+1) \\ &= 82.83(82^{20}+1)(82^{10}+1)(82^{5}-1)(82^{4}-82^{3}+82^{2}-82+1) \\ &= M(83). \end{aligned}$$

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153 Primes of the 10th Million:

by W. B. Davis, B.A.

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9260117
9357163
9422407
9524717
9705911
9745639
9768487
9820289
9826079
9880447
9890219
9949259
9963137

NOTE ON THE NUMBER OF PRIMES IN THE 7TH MILLION:

by W. B. DAVIS, B.A.

It is well known to readers of the Theory of Numbers that Legendre counted the number of Primes, in order to verify the log-integral formula, and I have since found, that Bessel in his *Briefwechsel* with Olbers, has done the same thing, verifying the general accuracy of Legendre's computation.

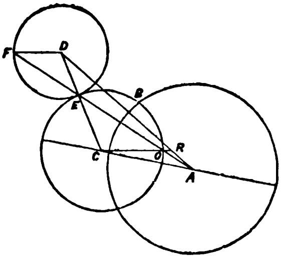
The following are the number of Primes in the successive 100,000, of the 7th Million. I suppress details.

1st	6386
2nd....	6401
3rd	6418
4th	6340
5th	6340
6th	6398
7th	6323
8th	6366
9th	6368
10th	6368

63708 Total number of primes in the 7th million; counted from Dase's *Factoren Tafeln.*

2330. (Proposed by H. TOMLINSON.)—Describe a circle whose centre shall be on a given diameter of a given circle, to cut that circle orthogonally and to touch another given circle.

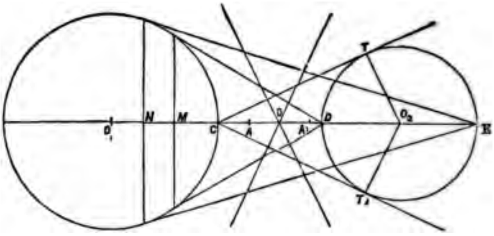
I. *Solution by* HUGH MURPHY.



Let the circle whose centre is A be that which is to be cut orthogonally, and the circle whose centre is D be that which is to be touched. Draw the lines as in the diagram: then AF . AE [= (tan)^2 from A to circle (D)], and AO . AE [= AB^2] are both given; therefore AF : AO or DF : RO is given, COR and DF being parallel; therefore RO is given; moreover the point R is given by the proportion AF : AO = AD : AR. Hence the question is reduced to the following one: given two points D and R to find a point C in a given line AC, so that (CD−CR) may be given. (See Mulcahy's *Modern Geometry,* p. 63, Note.)

Otherwise: as the envelope of the circle touching one fixed circle and cutting another orthogonally is a given circle, its centre and the centre of the touched circle are given points; hence the problem is reduced to finding a point in the given diameter, such that the difference of the lines drawn therefrom to these two given points may be equal to the difference of the radii.

II. *Solution by* James Dale.



1. To find the locus of the centre of a circle which shall cut one given circle orthogonally, and touch another given circle. Taking the origin of rectangular coordinates at the centre of the orthogonally cut circle, the line of centres being the axis of x, let r_1, r_2 be the radii, and d the distance of centres; then, if (x, y) be any point on the locus we have

$$(x^2+y^2-r_1^2)^{\frac{1}{2}} = \left\{(d-x)^2+y^2\right\}^{\frac{1}{2}} \pm r_2.$$

Clearing of radicals and arranging, this equation becomes

$$4(d^2-r_2^2)x^2-4r_2^2y^2-4d(r_1^2-r_2^2+d^2)x$$
$$+\left\{r_1^2+(r_2+d)^2\right\}\left\{r_1^2+(r_2-d)^2\right\}=0,$$

which represents an hyperbola, the equation of which, referred to the principal axes, is

$$4(d^2-r_2^2)x^2-4r_2^2y^2 = \frac{r_2^2\left\{d^2-(r_1^2+r_2^2)\right\}^2}{d^2-r_2^2}.$$

As a diameter of the circle (r_1) will in general cut the hyperbola in two points, two circles, having their centres respectively at the points of intersection, can be drawn cutting the one circle (r_1) orthogonally, and touching the other (r_2), the one externally, and the other internally.

The vertices and asymptotes of the hyperbola can be readily constructed as follows:—Let O_1, O_2 be the centres of the two circles, D, E the points in which the line O_1O_2 cuts the circle (O_2), then the vertices are the points of bisection of the distances between D, E and their polars with regard to the circle (O_1). Again, from C, the point of (O_1) nearest to O_2, draw tangents, CT, CT_1 to O_2, join O_2T and O_2T', then the asymptotes are parallel to these lines O_2T, O_2T'.

When $d^2=2r_2^2$, the hyperbola becomes equilateral; and when $d^2=r_1^2+r_2^2$, that is, when the circles O_1, O_2 cut orthogonally, the hyperbola becomes the two straight lines $r_1x+r_2y=0$ and $r_1x-r_2y=0$.

2227. (Proposed by W. Lea.)—Form 2^n-1 symbols into triads, so that every pair of the symbols shall appear once in the triads.

Solution by Samuel Bills.

The following is perhaps the simplest method of solution to this question. I shall give examples when there are 7, 15, and 31 symbols, and it will be seen that the method will be equally applicable for any value of n.

Take 7 symbols a, b, c, d, e, f, g, and form all the pairs in order as follows, *ab, ac, ad, ae, af, ag. bc, bd, be, bf, bg. cd, ce, cf, cg, de, df, dg, ef, eg, fg.* Now each triad will comprise three pairs; take the letters in the first two pairs for the first triad, viz., *abc*, and mark off the pairs of which it is composed with a dash underneath; then take the next pair in order, not marked off, and join it with the next succeeding one which will not comprise a pair that has been previously marked off, for the second triad, which will be *ade*, and mark off the three pairs as before. By proceeding in this manner till *all* the pairs are exhausted, we obtain the following seven triads,

abc, ade, afg, bdf, beg, cdg, cef.

Next take 15 symbols, *a, b, c, d, e, f, g, h, i, j, k, l, m, n, o,* and form the pairs in order thus:—
ab, ac, ad, ae, af, ag, ah, ai, aj, ak, al, am, an, ao, bc, bd, be, bf. bg, bh, bi, bj, bk, bl, bm, bn, bo, cd, ce, cf, cg, ch, ci, cj, ck, cl, cm, cn, co. de, df, dg, dh, di, dj, dk, dl, dm, dn, do, ef, eg. eh, ei, ej, ek, el, em, en, eo, fg, fh, fi, fj. fk, fl, fm, fn, fo, gh, gi, gj, gk, gl, gm, gn, go, hi, hj, hk, hl, hm, hn, ho, ij, ik, il, im, in, io, jk, jl, jm. jn, jo, kl, km, kn, ko, lm, ln, lo, mn, mo. no.

The number of pairs is 105, and by forming the triads from them as above described we obtain the following 35, which are put down exactly in the order in which they were obtained; *abc, ade, afg, ahi, ajk, alm, ano, bdf, beg, bhj, bik, bln, bmo, cdg, cef, chk, cij, clo, cmn, dhl, dim, djn, dko, ehn, eil, ejo, ekn, fhn, fio, fjl, fkm, gho, gin, gjm, gkl.*

In the above 35 triads *all* the pairs will be found. In precisely the same manner we may proceed for 31, 63, &c., symbols.

For the case of 31 symbols, let the symbols be $a, b, c, \ldots. y, z, a_1, b_1, c_1, d_1, e_1$; and form all the pairs in order as follows:—
$ab, ac, ad, \ldots. ae_1$; $bc, bd, \ldots. be_1$; $cd, ce_1 \ldots. ce_1$; $de, df, \ldots. de_1$; $ef, eg, .. ee_1$; $fy, fh, \ldots. fe_1$; $gh, gi, \ldots. ge_1$; $hi, hj \ldots. he_1$; $ij, ik, \ldots. ie_1$, $jk, jl, \ldots je_1$; $kl, km, \ldots. ke_1$; $lm, ln, \ldots. le_1$; $mn, mo, \ldots. me_1$; $no, np, \ldots. ne_1$; $op, oq, \ldots. oe_1$; $pq, pr, \ldots. pe_1$; $qr, qs, \ldots. qe_1$; $rs, rt, \ldots. re_1$; $st, su, \ldots. se_1$; $tu, tv, \ldots. te_1$; $nv, nw, \ldots. ne_1$; $vw, vx, \ldots. ve_1$; $wx, wg, \ldots. we_1$; $xy, xz, \ldots. xe_1$; $yz, ya_1, \ldots. ye_1$; $za_1, zb_1, \ldots. ze_1$; $a_1b_1, a_1c_1, \ldots. a_1e_1$; $b_1c_1, b_1d_1, b_1e_1, c_1d_1, c_1e_1, d_1e_1$.

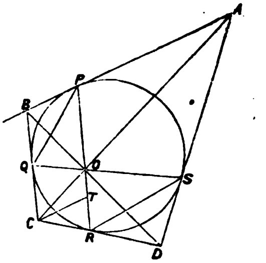
The complete list of pairs may be very readily written out as required.

Now by forming the triads from the pairs, as previously directed, we readily find the following 155 triads, which are put down in exactly the same order as they were obtained, and which will be found to exhaust *all* the pairs.
$abc, ade, afg, ahi, ajk, alm, ano, apq, ars, atu, avw, axy, aza_1, ab_1c_1, ad_1e_1, bdf, beg, bhj, bik, bln, bmo, bpr, bqs, btv, buw, bxz, bya_1, bb_1d_1, bc_1e_1, cdg, cef, chk, cij, clo, cmn, cps, cqr, ctw, cuv, cxa_1, cyz, cb_1e_1, cc_1d_1, dhl, dim, djn, dko, dht, dqu, drv, dsw, dxb_1, dyc_1, dzd_1, da_1e_1, ehn, eil, ejo, ekn, epu, eqt, erw, esv, exc_1, eyb_1, eze_1, ea_1d_1, fhn, fio, fjl, fkm, fpv, fqw, frt, fsu, fxd_1, fye_1, fzb_1, fa_1c_1, gho, gin, gjm, gkl, gpw, gqv, gru, gst, gxe_1, gyd_1, gzc_1, ga_1b_1, hpx, hgy, hrz, hsa_1, htb_1, huc_1, hvd_1, hwe_1, ipy, iqx, iva_1, isz, itc_1, iub_1, ive_1, iwd_1, jpz, jqa_1, jrx, jsy, jtd_1, jue_1, jvb_1, jwc_1, kpa_1, kqz, kry, ksx, kte_1, kud_1, kvc_1, kwb_1, lpb_1, lqc_1, lrd_1, lse_1, ltx, luy, lvz, lwa_1, mpc_1, mqb_1, mre_1, msd_1, mty, mux, mva_1, mwz, npd_1, nqe_1, nrb_1, nsc_1, ntz, nua_1, nvx, nwy, ope_1, oqd_1, orc_1, osb_1, ota_1, ouz, ovy, owx.$

2310. (Proposed by J. M. Wilson, M.A., F.G.S.)—Prove (1) that if ABCD be a quadrilateral circumscribed about a circle, and P, Q, R, S the points of contact, then AC, BD, RP, SQ pass through one point.

Hence show (2) that the locus of the intersection of the lines joining the ends of chords in a circle which pass through a given point is the polar of that point.

I. *Solution by the* PROPOSER.



1. Consider the segments into which PR divides the diagonal AC.

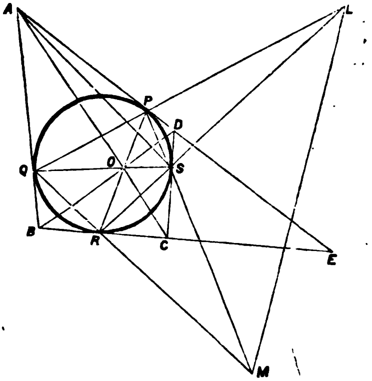
Through C draw a line CT parallel to AP, then CT = CR and AO : OC = AP : CR.

Similarly QS divides AC into segments which have the ratio of AS to QC, that is of AP to CR. Hence PR and QS intersect on the diagonal AC, and therefore also on BD.

2. To prove the second part, join PQ, RS. Then B is the pole of PQ, D of SR, and therefore the intersection of PQ, SR is the pole of BD which passes through O, and therefore PQ and SR intersect on the polar of O.

NOTE.—These well-known properties are proved in McDowell's *Exercises*, by the harmonic properties of transversals to a triangle, and of chords of a circle.

II. *General Solution by the* REV. R. H. WRIGHT, M.A.; *the* REV. J. L. KITCHIN, M.A.; H. TOMLINSON; W. CHADWICK; *and others.*



Let ABE be a triangle of reference where BE, EA, AB are respectively represented by $\alpha = 0$, $\beta = 0$, $\gamma = 0$; also let PQR be an inscribed conic; and draw any tangent DC touching the conic in S. Now if the equation of the inscribed conic be $(l\alpha)^{\frac{1}{2}} + (m\beta)^{\frac{1}{2}} + (n\gamma)^{\frac{1}{2}} = 0$, and if AS be represented by $\beta - h\gamma = 0$, the equation of CD is (Wright's *Examples on Trilinears*, Art. 74.)

$$lh^{\frac{1}{2}}\alpha - (m^{\frac{1}{2}}h^{\frac{1}{2}} + n^{\frac{1}{2}})\, m^{\frac{1}{2}}\beta - (m^{\frac{1}{2}}h^{\frac{1}{2}} + n^{\frac{1}{2}})\, n^{\frac{1}{2}}h^{\frac{1}{2}}\gamma = 0 \quad \ldots\ldots\ldots\ldots (1).$$

Making $\beta = 0$ and $\alpha = 0$ respectively in (1), we have

for BD and AC.... $(m^{\frac{1}{2}}h^{\frac{1}{2}} + n^{\frac{1}{2}})\, n^{\frac{1}{2}}\gamma - l\alpha = 0$, and $m^{\frac{1}{2}}\beta + n^{\frac{1}{2}}h^{\frac{1}{2}}\gamma = 0$.. (2, 3),
and we readily get
for PR.................. $l\alpha + m\beta - n\gamma = 0$ .... (4),
and QS.... $lm^{\frac{1}{2}}\alpha - (m^{\frac{1}{2}}h^{\frac{1}{2}} + n^{\frac{1}{2}})\, m^{\frac{1}{2}}n^{\frac{1}{2}}\beta - (m^{\frac{1}{2}}h^{\frac{1}{2}} + n^{\frac{1}{2}})\, (n^{\frac{1}{2}}h^{\frac{1}{2}} + m^{\frac{1}{2}})\, n^{\frac{1}{2}}\gamma = 0$ .. (5).

Now $(n^{\frac{1}{2}}h^{\frac{1}{2}} + m^{\frac{1}{2}})\,(2) + n\,(3) + n^{\frac{1}{2}}h^{\frac{1}{2}}\,(4) + (5) \equiv 0$,
which shows that AC, BD, RP, SQ pass through the same point.

2. If we refer to the *Mathematician*, (Vol. I., p. 133), we find that if the sides PS and QR of the quadrilateral PQRS inscribed in the conic be produced to meet in M, and QP and RS to meet in L, and if the diagonals meet in O; then ML is the polar of O, and OL is the polar of M.

NOTE ON A LOCUS CONNECTED WITH THE GEOMETRY OF THE TRIANGLE.

BY J. GRIFFITHS, M.A.

The locus referrad to is that of a point (ξ, η, ζ) such that its pedal circle* touches the nine-point circle of the triangle of reference. Its equation is,

$$\Sigma\left\{\xi \cos^2 A\,(\eta \cos B - \zeta \cos C)\,(\eta \cos C - \zeta \cos B)\right\}^{\frac{1}{2}} = 0,$$

and the object of the present note is to show that the curve, though apparently a proper one of the sixth degree, consists in fact of two coincident *cubics* given by the common equation

$$\xi \cos A\,(\eta^2 - \zeta^2) + \eta \cos B\,(\zeta^2 - \xi^2) + \zeta \cos C\,(\xi^2 - \eta^2) = 0.$$

For if we write

$$\cos A\,(\eta \cos C - \zeta \cos B) = x, \quad \cos B\,(\zeta \cos A - \xi \cos C) = y,$$
$$\cos C\,(\xi \cos B - \eta \cos A) = z; \quad \xi \cos A\,(\eta \cos B - \zeta \cos C) = x',$$
$$\eta \cos B\,(\zeta \cos C - \xi \cos A) = y', \quad \zeta \cos C\,(\xi \cos A - \eta \cos B) = z',$$

the above equation becomes $(xx')^{\frac{1}{2}} + (yy')^{\frac{1}{2}} + (zz')^{\frac{1}{2}} = 0$,
where $x + y + z = 0$, and $x' + y' + z' = 0$.

Hence, eliminating z and z', for instance, we easily find $(xy' - x y)^2 = 0$, or

$$\cos A\,(\eta \cos C - \zeta \cos B)\,\eta \cos B\,(\zeta \cos C - \xi \cos A)$$
$$= \xi \cos A\,(\eta \cos B - \zeta \cos C)\cos B\,(\zeta \cos A - \xi \cos C),$$

that is, $$\Sigma\,\xi \cos A\,(\eta^2 - \zeta^2) = 0.$$

Hence we see that the pedal of (ξ, η, ζ) touches the nine-point circle, provided that the line joining (ξ, η, ζ) to its inverse $(\xi^{-1}, \eta^{-1}, \zeta^{-1})$ passes through the centre of the circumscribing circle.

AT RANDOM: No. V. By PROFESSOR WHITWORTH.

As Mr. Woolhouse still adheres to his assertion, that the phrase "at random," has in general only one absolute and unqualified meaning, perhaps I may be allowed to illustrate the question by reference to Mr. Woolhouse's own language in the statement of the last two problems which he has published in the *Educational Times*.

His Question (2420) in the June number, commences as follows :—"Three numbers, each of them not exceeding a number n, being arbitrarily taken, &c." This may mean, either that a *group* is selected arbitrarily (any *group* being equally likely), or else that the numbers are severally selected arbitrarily, in which case the group 2, 3, 4 will be *six* times as likely to occur, and the group 3, 3, 4 will be *three* times as likely to occur as the group 3, 3, 3. It seems to me, that Mr. Woolhouse's language, in the statement of his question, is more suggestive of the latter interpretation; but the result which he arrives at shows that he intends the former. If he had used an absolute verb instead of a participle, the distinction might easily have been expressed. The first case would have been indicated by the phrase, "A group of three numbers *is* arbitrarily taken;" the second by the phrase, "Three numbers are arbitrarily taken." In the one case, we have to make *one* selection of *three* numbers; in the other, *three* selections of *one* number.

[* That is, the circle through the feet of the perpendiculars from the point (ξ, η, ζ) on the sides of the triangle of reference. This circle also passes through the feet of the perpendiculars from the $(\xi^{-1}, \eta^{-1}, \zeta^{-1})$ on the sides, see solutions of Quests. 1815 and 1975, *Reprint*, Vol. V .p. 19, and Vol. VI. p. 43.]

In the one case, all *combinations* (repetitions admissible) are equally likely: in the other, all *permutations* (repetitions admissible) are equally likely.

A precisely similar ambiguity occurs in Mr. Woolhouse's Question (2433) in the July Number. Does he mean, that five points *are* selected at random, or that a selection of five points *is* made at random?

But as to Mr. Woolhouse's general assertion, that the phrase "at random" must be unambiguous, having only one proper meaning in any enunciation, I can only say that I, for one, agree with him if he means that by due care we can generally so express ourselves as to preclude any doubt about the meaning of our most unqualified statements; but if he means that such loose language as we are in the habit of meeting with, his own Question 2420 affording an instance, is never open to a double interpretation, his appreciation of the niceties of language must be much more precise than mine.

2371. (Proposed by Professor CAYLEY.)—(4). If P, Q be two points taken at random within the triangle ABC, what is the chance that the points A, B, P, Q may form a convex quadrangle?

Solution by PROFESSOR SYLVESTER.

1. If the quadrangle ABPQ is non-convex, ABQ encloses P or ABP encloses Q, and therefore the chance of a reentrant figure is double the chance of ABP containing Q.

Divide the triangle into infinitesimal bands parallel to AB; then taking any element of given magnitude in this band, the chance of P being upon it and Q inside the triangle APB is the same all along the band and is proportonal to the distance of the band from the base. Hence if the distance of the apex from the base is h, and from the band x (remembering that the length of the band is proportional to x, and its distance from the base to $h-x$) the total chance of APB enclosing Q is

$$\int_0^h dx\,(hx-x^2) \div \int_0^h dx\,.\,hx = \frac{\frac{1}{2}-\frac{1}{3}}{\frac{1}{2}} = \tfrac{1}{3}.$$

Hence the chance of the figure APQB being reentrant is $\frac{2}{3}$, and of being convex is $\frac{1}{3}$, being independent of the form of the triangle.

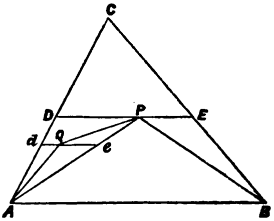
2. In like manner it may easily be shown that for a parallelogram on AB the chance of the quandrangle APQB being reentrant is

$$2\int_0^h dx\,.\,\frac{x}{2} \div \int_0^h dx\,.\,h = \tfrac{1}{2}.$$

Hence it is an even chance whether the quadrangle is convex or reentrant. The principle of duplicity above employed is one of frequent use in the calculus of local probability; by aid of it in Professor CAYLEY'S question we have been enabled to elude the calculation of the troublesome integral $\int dm$ (CMPN), M and N being the points where AP and BP meet the opposite sides, as we know à priori that this must be equal to $\int dm\,.\,$APB.

3. The above method applies with still more conspicuous advantage to an extension of Professor Cayley's problem to *five* points. Suppose we want to

calculate the probability of ABPQR forming a convex figure; P, Q, R being any three points inside the triangle ABC.



If the pentagon is not convex, either two of the points P, Q, R will be within the triangle formed by the third with AB, or one of them will lie inside the quadrangle formed by the remaining two with AB.

The chance of the former is three times the chance of two specified points Q, R lying within the triangle ABP.

The chance of the latter is three times the chance of one specified point R lying within the quadrangle ABPQ; or if we please so to say, is six times the chance of this taking place coupled with the new condition of Q being nearer than P is to the base AB.

If then we draw a band parallel to AB through P, the latter chance will be twelve times the chance of Q lying in APD whilst R lies in APB or AQP, the chances relating to Q placed in APD and to Q placed in BPE being of course the same.

If then the chance of Q, R being each of them inside APB be called ϕ,
that of Q being in APD and R in APB be called ψ,
that of Q being in APD and R in AQP be called ω,
the chance of the figure APQRB being convex will be

$$1-3\phi-12\psi-12\omega.$$

It is obvious *a priori* that the form of the triangle does not enter into the result. Call then the perpendicular distance of C from the base and the length of the latter each unity, so that the area of ABC $= \frac{1}{2}$, and let the triangle APD be divided into bands *d e* parallel to AB or DE.

Let x be the distance of DE from AB, ξ that of *de* from AB, and put DP $= y$. Then since the cube of the given triangle is $\frac{1}{8}$, we have

$$\tfrac{1}{8}\phi = \int_0^1 dx\,(1-x)\left(\frac{x}{2}\right)^2 = \tfrac{1}{4}\left(\tfrac{1}{3}-\tfrac{1}{4}\right) = \tfrac{1}{48};$$

$$\tfrac{1}{8}\psi = \int_0^1 dx \int_0^{1-x} dy\,.\,y\left(\frac{x}{2}\right)^2 = \int_0^1 dx\,(1-x)^2\,\frac{x^2}{8} = \tfrac{1}{240};$$

$$\tfrac{1}{8}\omega = \int_0^1 dx \int_0^{1-x} dy \int_0^x d\xi\left(\frac{\xi y}{x}\right)^2 \frac{x}{4} = \int_0^1 dx\,(1-x)^3\,\frac{x^2}{36} = \tfrac{1}{2160}.$$

Hence the required chance of the figure APQRB being convex is

$$1-\tfrac{1}{2}-\tfrac{2}{5}-\tfrac{2}{45}, \text{ or } \tfrac{1}{18}.$$

In analysis, as in statecraft, the great rule applies, "Divide et impera.'

4. The same method gives a simple solution of the four-point problem when the four points are taken anywhere inside a given triangle.

The chance of their forming a reentrant figure is four times that of a specified one of them (as S) lying within the triangle whose apices are the other three (P, Q, R). Now the mean value of the triangle PQR is one-twelfth of the limiting triangle, as is shown in the solutions of Quest. 1229, (*Reprint*, Vol. II., p. 95, and Vol. IV., p. 101; in the *first* line of the latter solution, however, m is printed in error for M). Hence the chance of S being within

PQR is $\frac{1}{12}$; and the whole chance of the figure PQRS being reentrant is therefore $\frac{1}{3}$; so that the complementary chance of the four points forming a convex quadrangle is $\frac{2}{3}$.

5. It is worth while to show here how the result of Art. 4 may be obtained more directly by another mode of applying the theorem (Ω) in the passage from the *Reprint*, Vol. IV., last referred to.

We may make n (the number of points) equal to 4, i (the order of the *affection*) equal to zero; we have then

$$M = \frac{8.4.3.2}{8.7.6} m + \frac{4.4.3}{8.7.6}(3\mu) = \frac{4m+3\mu}{7}$$

where $m = \frac{1}{4}$, and μ = the double of $\frac{2}{9} = \frac{4}{9}$. Hence $M = \frac{3+4}{21} = \frac{1}{3}$, the probability, as previously found, of four points inside a given triangle forming the apices of a reentrant quadrangle.

6. Let us apply the same method to determining the probability of five points taken anywhere at random within the triangle ABC forming or not a convex figure. It may easily be seen *à priori* that the relative lengths of the sides will not affect the result, and I shall regard them all as *units*. Call m the probability of *reentrance* when three of the five points lie respectively along three sides, μ the same when two of them lie respectively at a vertex and on an opposite side of the limiting triangle. Then, if the required chance of reentrance is M, we have by (Ω)

$$M = \frac{8.5.4.3\,m + 4.5.4(3\mu)}{10.9.8} = \frac{2m+\mu}{3}.$$

(α). To find μ; calling B the fixed vertex, Q the point moveable in AC, we have

$$\mu = \frac{17}{18}\mu_1 + \frac{2}{3}\mu_2,$$

where μ_1 = mean value of $\frac{(ABQ)^3 + (CBQ)^3}{(ABC)^3}$,

and μ_2 = mean value of $\frac{3(ABQ)^2(CBQ) + 3(ABQ)(CBQ)^2}{(ABC)^3}$.

Hence $\mu_1 = 2\int_0^1 dx\,.\,x^3 = \frac{1}{2}$; $\mu_2 = 6\int_0^1 dx\,(x^2 - x^3) = \frac{1}{2}$;

therefore $$\mu = \frac{17}{36} + \frac{1}{3} = \frac{29}{36}.$$

(β). Again, P, Q, R being points taken at random in BC, CA, AB respectively, we have $m = m_1 - m_2 + \frac{2}{3} m_3$,

where m_1 = mean value of $2\frac{PQR}{ABC}$,

also m_2 = mean value of $\frac{(PQR)^2}{(ABC)^2}$,

and m_3 = mean value of $\Sigma\frac{(PBR)^2}{(ABC)^2}$, that is of $3\frac{(PBR)^2}{(ABC)^2}$.

where it is to be noticed that $\frac{2}{3}$ is the coefficient of reentrance given in Art. (1), and that the mean values of such products as $\frac{(\text{PBR})\,(\text{QCP})}{(\text{ABC})^2}$ do not enter into m because the probabilities due to the *distribution* of the two outstanding points *between* PBR, QCB is *nil.*

The mean value of PQR is easily seen to be the triangle whose apices are the mid-points of the sides of ABC.

Hence we have $\frac{m_1}{2} = \frac{1}{4}$ or $m_1 = \frac{1}{2}$.

To find m_2; for the moment suppose P and R fixed whilst Q travels along AC; make $\text{AQ} = x$, $\text{ARP} = \alpha$, and $\text{CPR} = \beta$.
Then the mean value of $(\text{PQR})^2$ becomes

$$\int_0^1 dx \left\{(1-x)\,\alpha + x\beta\right\}^2 = \frac{1}{3}(\alpha^2 + \alpha\beta + \beta^2).$$

The mean value of α^2 (as of β^2) is $\left(\int_0^1 dx\,.\,x^2\right)^2(\text{ABC})^2$, i.e. $\frac{(\text{ABC})^2}{9}$.

The mean value of $\alpha\beta$ is $\left(\int_0^1 dx\,(x-x^2)\right)^2(\text{ABC})^2$, i.e. $\frac{(\text{ABC})^2}{36}$;

where in explanation of the *latter* equation it may be observed that when R remains fixed α varies as BP, say x, whilst β varies as CP or $(1-x)$, so that the mean value of $\alpha\beta$ on this preliminary supposition varies as

$$\int_0^1 dx\,.\,x\,(1-x)\,(\text{ACR}\,.\,\text{BCR}).$$

Hence we have $m_2 = \frac{1}{3}\left(\frac{1}{9} + \frac{1}{36} + \frac{1}{9}\right) = \frac{1}{12}$.

Finally the mean value of $\frac{(\text{PBR})^2}{(\text{ABC})^2} = \left(\int_0^1 dx\,.\,x^2\right)^2 = \frac{1}{9}$, so that $m_3 = \frac{2}{9}$.

Thus $m = \frac{1}{2} - \frac{1}{12} + \frac{2}{9} = \frac{23}{36}$ and $\text{M} = \frac{1}{3}\left(\frac{46+29}{36}\right) = \frac{25}{36}$, which fraction is the chance of reentrance for 5 points taken at random in *any* given triangle, thus giving $\frac{11}{36}$ as the chance of convexity.

7. The reentrance referred to above may be single or double. The calculation for the case of double reentrance isolated from that of single reentrance is very easy. Following the notation in Art. (1), the chance of three points in any triangle ABC forming with A and B the apices of a doubly reentrant pentagon is three times

$$\int_0^h x\,(h-x)^2\,.\,dx \div \int_0^h xh^2\,.\,dx, \text{ that is } \frac{1}{2}.$$

Then following the notation of Art. 6, so that μ_1, μ_2, m_1, m_2, m_3 remain the same as before, we have now

$$\mu = \frac{1}{2}\mu_1 + 0 \,.\, \mu_2 = \frac{1}{4}, \text{ and } m = m_2 = \frac{1}{12};$$

therefore $$M = \frac{1}{3}\left(\frac{2}{12} + \frac{1}{4}\right) = \frac{5}{36}.$$

Mr. WOOLHOUSE's formula (Quest. 2433) gives for this case

$$M = \frac{5}{12} \cdot \frac{(2 + \cos \frac{2}{3}\pi)^2}{(3 \sin \frac{2}{3}\pi)^2} = \frac{5}{12} \cdot \frac{1}{3} = \frac{5}{36}, \text{ as found above.}$$

Thus the probabilities of the pentagon cornered by five points taken at random on any triangle being doubly concave, simply concave, and totally convex, will be as the numbers 5, 20, 11.

8. The value of the *total* chances of concavity calculated in Art 3, by aid of the diagram therein given may easily be seen to be

$$\int_0^1 dx \int_0^{1-x} dy \,.\, 2\left(3x^2 + 12x^2y + 6 \,.\, \frac{2}{3}\, x^2y^2\right).$$

The first term $6x^2$ within the brackets gives as the final result of the integration $\frac{1}{2}$, the chance calculated in Art. 2.

The second term $24x^2y$ gives $12\left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5}\right) = \frac{2}{5}$.

The third term $8x^2y^2$ gives $\frac{8}{3}\left(\frac{1}{3} - \frac{3}{4} + \frac{3}{5} - \frac{1}{6}\right) = \frac{2}{45}$.

Now $\frac{2}{5} + \frac{2}{45} = \frac{4}{9}$, which is the chance of simple reentrance for the pentagon whose angles are A, B, and three points taken at random within ABC; and $\frac{1}{2} + \frac{4}{9} = \frac{17}{18}$ which confirms the correctness of the result found in Art. 3.

9. There is no difficulty in finding the chance that of $\nu+2$ points, taken at random in a triangle, $\nu-1$ of them shall be within the remaining three. This however it should be observed when $\nu > 3$ is not the case of a simple "morph", but comprises various distinguishable species of configuration. Beginning with the supposition of two of the points lying on given angles of the triangle, this collective chance will easily be found to be

$$2\nu \int_0^1 dx\, x^{\nu-1}(1-x) = \frac{2}{\nu+1}.$$

Then passing to the case of all $\nu+1$ points being free, we obtain

$$\mu = \frac{4}{\nu+1}\int_0^1 dx.x^{\nu} = \frac{4}{(\nu+1)^2},$$

$$m = \text{mean value of } \left(\frac{PQR}{ABC}\right)^{\nu}$$

$$= \text{mean value of } \frac{1}{\nu.(ABC)^{\nu}}\left(\alpha^{\nu-1} + \alpha^{\nu-2}\beta + \alpha^{\nu-3}\beta^2 + \ldots\ldots + \beta^{n-1}\right)$$

$$=\frac{1}{\nu^3}\left\{1+(\nu-1)^{-2}+\left[(\nu-1)\left(\frac{\nu-2}{2}\right)\right]^{-2}+\left[(\nu-1)\left(\frac{\nu-2}{2}\right)\left(\frac{\nu-3}{3}\right)\right]^{-2}+.+1\right\}$$

Finally $$M = \frac{8\nu(\nu+1)(\nu+2)m+12(\nu+1)(\nu+2)\mu}{(2\nu+2)(2\nu+3)(2\nu+4)}$$

$$=\frac{2}{2\nu+3}\left\{\frac{s}{\nu^2}+\frac{6}{(\nu+1)^2}\right\},$$

where s is the sum of the squared reciprocals of the coefficients in the development of $(1+t)^{\nu-1}$ according to the powers of t.*

1917. (Proposed by W. S. B. WOOLHOUSE, F.R.A.S.)—Suppose the surface of a sphere to be made up of an indefinite number of points, and straight lines to be drawn through every two of those points, and determine the law of density of this mass of lines as depending on the distance from the centre of the sphere.

Solution by the PROPOSER.

Let a plane be drawn through the centre of the sphere, and first consider the system of parallel chords that are normal to this plane. To simplify our notions, suppose the plane to be horizontal and the chords under consideration to be therefore vertical.

Let a denote the radius of the sphere;
z the singular zenith distance of any point on the surface;
n the number of points, supposed to be indefinitely great and uniformly distributed over the surface;
ρ the radius of a sphere concentric with the given one;
V its volume;
L the quantity of line imparted to the volume of this concentric sphere by the mass of chords joining every pair of the n points;
D the density of the mass of lines at the distance ρ from the centre;
$k=\frac{\rho}{a}$, and $\lambda=\frac{n^2}{8\pi a}$.

Then, conceiving two consecutive horizontal parallel circles whose angular zenith distances are z, $z+dz$, the spherical radii of these circles are az, $a(z+dz)$, and the area of the spherical surface contained between them is $2\pi a\sin z\times adz = 2\pi a^2 dz\sin z$. And as the n points are uniformly distributed over the total spherical area $4\pi a^2\int dz\sin z = 4\pi a^2$, the proportionate number of points contained between these parallel circles is $\frac{2\pi a^2 dz\sin z}{4\pi a^2}n$ $=\frac{1}{2}ndz\sin z$. The vertical chords which connect these points with the corresponding points in the lower hemisphere generate a concentric cylindrical annulus, radii $a\sin z$, $a\sin(z+dz)$ and altitude $2a\cos z$, and the portion of each of these vertical chords which lies within the concentric sphere, radius ρ, volume V, is evidently $2\sqrt{(\rho^2-a^2\sin^2 z)}$. Hence, differentiating with respect to ρ, we get the sum of the two equal portions which fall within the differential volume dV contained between two consecutive concentric spheres, radii ρ, $\rho+d\rho$, equal to $\dfrac{2\rho d\rho}{\sqrt{(\rho^2-a^2\sin^2 z)}}$.

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* Thus, making ν successively 1, 2, 3, M assumes the values 1, $\frac{1}{3}$, $\frac{5}{36}$, which is right.

Now if the sphere be moveable round its centre, and each of the n points on the surface be made a zenith, and a set of vertical parallel chords drawn, we shall evidently obtain, in the aggregate, the whole of the chords which connect every two points on the surface; and as these n sets of parallel chords are precisely alike, they will all be included by simply multiplying by n. From these considerations we find that the quantity of line thrown into the differential volume dV by the whole mass of chords is

$$d\mathrm{L} = n\times\int\frac{n}{2}\,dz\sin z\,.\,\frac{2\rho d\rho}{\sqrt{(\rho^2-a^2\sin^2 z)}} = n^2\rho d\rho\int\frac{dz\sin z}{\sqrt{(\rho^2-a^2\sin^2 z)}}$$

$$= \frac{n^2\rho d\rho}{a}\log\frac{a\cos z-\sqrt{(\rho^2-a^2\sin^2 z)}}{a-\rho}\ (\text{up to } a\sin z=\rho)$$

$$= \frac{n^2\rho d\rho}{2a}\log\frac{a+\rho}{a-\rho}.$$

And as this quantity of line is obviously equally distributed round the differential volume, $d\mathrm{V} = 4\pi\rho^2 d\rho$, the density of the mass of lines round this spherical annulus at the distance ρ from the centre is

$$\mathrm{D} = \frac{d\mathrm{L}}{d\mathrm{V}} = \frac{n^2}{8\pi a\rho}\log\frac{a+\rho}{a-\rho} = \frac{\lambda}{\rho}\log\frac{a+\rho}{a-\rho},$$

which is the required law of density.

When developed in a series it is

$$\mathrm{D} = \frac{\lambda}{\rho}\int\frac{2a d\rho}{a^2-\rho^2} = \frac{2\lambda}{a\rho}\int\frac{d\rho}{1-\frac{\rho^2}{a^2}} = \frac{2\lambda}{a}\left(1+\frac{\rho^2}{3a^2}+\frac{\rho^4}{5a^4}+\&c.\right)$$

$$= \frac{n^2}{4\pi a^2}\left(1+\frac{\rho^2}{3a^2}+\frac{\rho^4}{5a^4}+\&c.\right)$$

And at the centre $\mathrm{D}_0 = \dfrac{2\lambda}{a} = \dfrac{n^2}{4\pi a^2}$.

So that, putting $\dfrac{\rho}{a} = k$, the law of density may be stated thus:

$$\frac{\mathrm{D}}{\mathrm{D}_0} = \frac{1}{2k}\log\frac{1+k}{1-k}.$$

A small marginal table is here annexed to show the numerical character of this function, which expresses the relative density at the distance k from the centre when the radius is taken as the linear unity.

k	$\dfrac{\mathrm{D}}{\mathrm{D}_0}$
·0	1·0000
·1	1·0033
·2	1·0137
·3	1·0317
·4	1·0591
·5	1·0986
·6	1·1552
·7	1·2390
·8	1·3733
·9	1·6358
1·0	∞

The quantity of line within the sphere (ρ) is found by integrating

$$d\mathrm{L} = \frac{n^2}{2a}\rho d\rho\log\frac{a+\rho}{a-\rho},\quad\text{and is } \mathrm{L} = \frac{n^2\rho}{2}\left(1-\frac{a^2-\rho^2}{2a\rho}\log\frac{a+\rho}{a-\rho}\right).$$

When $\rho = a$, we have the whole quantity of line contained within the given sphere $= \frac{1}{2}n^2 a = 2\pi a^3\,.\mathrm{D}_0$. Hence, as the number of chords is $\frac{1}{2}n^2$, the average length of all the chords is equal to the radius of the given sphere. Also, dividing by the total volume $\frac{4}{3}\pi a^3$, the average density for the whole sphere is $\frac{3}{2}D_0$. And if all the lines were uniformly distributed with the same density as that at the centre, they would occupy a volume $(2\pi a^3)$ which is equal to that of the cylinder which circumscribes the given sphere.

It will be observed that the density of the lines becomes infinite at the surface of the given sphere. If the chords be all supposed to be produced indefinitely outside the sphere, the external law of density is readily found from the foregoing expressions. In this case $\rho > a$, and in determining the value of dL, we have only to give to z its full limit $\frac{1}{2}\pi$. Thus we find

$$dL = \frac{n^2\rho d\rho}{2a}\log\frac{\rho+a}{\rho-a}, \text{ and hence } D = \frac{\lambda}{\rho}\log\frac{\rho+a}{\rho-a};$$

and, putting as before $\frac{\rho}{a} = k$, we get $\frac{D}{D_0} = \frac{1}{2k}\log\frac{k+1}{k-1}$.

These forms are remarkably analogous to the expressions we have found for the interior of the sphere, but they materially differ in their character. Thus, by expanding the interior and exterior densities in terms of k, we find

when $\quad k<1,\ \frac{D}{D_0} = 1 + \frac{k^2}{3} + \frac{k^4}{5} + \&c.$

when $\quad k>1,\ \frac{D}{D_0} = \frac{1}{k^2}\left(1 + \frac{1}{3k^2} + \frac{1}{5k^4} + \&c.\right)$

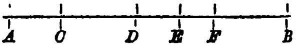
The latter law for remote distances virtually becomes inversely as the square of the distance. Also for reciprocal pairs of points we have the simple relation $D_{\frac{1}{k}} = k^2 D_k$, whatever be the value of k.

NOTE.—An analogous problem for the circle instead of a sphere was proposed by me as the Prize Question in the *Lady's and Gentleman's Diary* for 1866. The law of density for the circle is an elliptic function, and we have found that for the sphere to be a logarithmic function; but the resulting numerical law for the two cases presents a remarkable similarity. The increase of density from the centre outwards is somewhat greater for the sphere, and both of them become infinite on attaining the boundary.

On taking a finite number of points, and making a diagram for the circle, the beautiful clusters of intersections were so striking that I was induced to prepare diagrams on stone, impressions of which are published in the *Diary* for the present year. I have also had photographs taken of a very large one, constructed from 144 points, with a diameter of 17 inches, containing 10,296 connecting lines, and involving 17,178,876 intersections, for which extraordinary production I am indebted to the assiduity and perseverance of my esteemed friend, Mr. John Scott, of Lee.

1910. (Proposed by Professor SYLVESTER.)—AB is a given straight line upon which four points are taken at random. Find the chance that their anharmonic ratio (estimated by the quotient of the whole into the middle by the product of the extreme segments) shall exceed a given quantity.

Solution by Stephen Watson.



Let C, D, E, F be the points, and put AC $= w$, CD $= x$, DE $= y$, CF $= z$, AB $= a$, and m = the given quantity.

Then $$\frac{zy}{x(z-x-y)} > m, \text{ therefore } y > \frac{mx(z-x)}{mx+z} (= r \text{ suppose}),$$

and obviously y must be $< z-x$; also the limits of x, z, w are respectively $(0, z)$, $(0, a-w)$ and $(0, a)$; moreover the four points can be interchanged in 24 ways, and the measure of the total number of their positions is a^4; hence the required chance is

$$\frac{24}{a^4}\int_0^a dw \int_0^{a-w} dz \int_0^z dx \int_r^{z-x} dy = \frac{24}{a^4}\int_0^a dw \int_0^{a-w} dz \int_0^z \frac{z(z-x)\,dx}{mx+z}$$

$$= \frac{24}{a^4}\int_0^a dw \int_0^{a-w} \left\{\frac{m+1}{m^2}\log(m+1) - \frac{1}{m}\right\} z^2 dz$$

$$= 2\left\{\frac{m+1}{m^2}\log(m+1) - \frac{1}{m}\right\}.$$

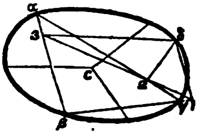
[Professor Sylvester remarks that two of the points may be fixed at A and B, by virtue of a general principle: hence the required chance becomes

$$2\int_0^1 \frac{1-x}{1+mx}\,dx = 2\left\{\frac{m+1}{m^2}\log(m+1) - \frac{1}{m}\right\}.$$

Thus the chance of the harmonic ratio being exceeded, that is of the ratio being ultra-harmonic, is 4 log 2−2, that is ·77258872, or $\frac{17}{22}$ nearly.]

2395. (Proposed by J. J. Walker, M.A.)—A proof is required of the following generalization of a well-known theorem:—If from any point on a conic lines be drawn to meet the three sides of any inscribed triangle, parallel to the diameters bisecting those sides respectively, the three points of intersection will be in the same straight line.

I. *Solution by* W. S. McCay; S. Bills; *and others.*



Let the eccentric angles of the vertices of the inscribed triangle be (α, β, γ), and δ that of any other point on the ellipse.

Then the equation of the side $\beta\gamma$ is

$$\frac{x}{a}\cos\frac{\beta+\gamma}{2} + \frac{y}{b}\sin\frac{\beta+\gamma}{2} = \cos\frac{\beta-\gamma}{2} \ldots\ldots\ldots\ldots\ldots (1).$$

Now the relation between the angles that two conjugate diameters make with the axis being $\tan\theta \tan\theta' = -\frac{b^2}{a^2}$, the equation of the line through δ parallel to the diameter bisecting (1) is easily seen to be

$$\frac{x}{a}\sin\frac{\beta+\gamma}{2} - \frac{y}{b}\cos\frac{\beta+\gamma}{2} = \sin\left(\frac{\beta+\gamma}{2}-\delta\right) \quad \ldots\ldots\ldots\ldots(2).$$

The intersection of (1) and (2) is one of the three points that are to be proved collinear; and the theorem would be proved if we could find a line through the intersection of (1) and (2) symmetrical with regard to (α, β, γ). Multiplying (1) by $\sin\frac{1}{2}(\alpha-\delta)$, (2) by $\cos\frac{1}{2}(\alpha-\delta)$ and subtracting, we get, after reducing the right hand side, the equation of a symmetrical line, viz.,

$$\frac{x}{a}\sin\frac{\alpha+\beta+\gamma-\delta}{2} - \frac{y}{b}\cos\frac{\alpha+\beta+\gamma-\delta}{2}$$

$$= 2\sin\frac{\alpha-\delta}{2}\sin\frac{\beta-\delta}{2}\sin\frac{\gamma-\delta}{2} + \sin\frac{\alpha+\beta+\gamma-3\delta}{2}.$$

II. *Solution by* MORGAN JENKINS, B.A.

The theorem may be generalized still further thus :—

Let ABC be a triangle inscribed in a conic; A′B′C′ the triangle circumscribed to the conic at the points A, B, C. Then if any three concurrent straight lines RA′, RB′, RC′ meet the polar of R in A″, B″ and C″ respectively, and straight lines through A″, B″, C″ respectively to any point T on the conic, meet BC, CA, AB respectively in α, β, γ; α, β, γ shall be collinear.

By making R the centre of the conic we obtain the theorem in the question.

Let BC, CA, AB meet the polar of R in α', β', γ' respectively,

TA, TB, TC meet the polar of R in α'', β'', γ'' respectively,

and the conic meet the polar of R in P, Q.

Then since α', &c., are the poles of A′A″, &c. respectively,

α', A″ ; β', B″ ; γ', C″ are corresponding points of a system in involution of which P, Q are foci ....(1);

and, since the polar of R cuts the inscribed quadrilateral TABC and the conic, therefore α', α''; β', β''; γ', γ''; P, Q are corresponding points of a system in involution........................ (2).

Hence A″, α''; B″, β''; C″, γ'' are corresponding points of another system in involution ....(3).

[We readily obtain (3) from (1) and (2); for if α', β',.... divide PQ in the ratio μ', ν'... respectively; A″, B″... in the ratios D, E... respectively; and α'', β''.. in the ratios μ'', ν''...... ; then

from (1) we have $D = -\mu'$, $E = -\nu'$, $F = -\rho'$;

from (2) we have $\mu'\mu'' = \nu'\nu'' = \rho'\rho''$;

hence $D\mu'' = E\nu'' = F\rho''$; whence (3) follows].

Therefore $A''\beta''.B''\gamma.C''\alpha'' = A''\gamma''.B''\alpha''.C''\beta''$.

But if F be a perpendicular from T on $\alpha\beta\gamma$

$$(A''\beta'')\,t = 2\frac{T\beta''}{TB}.(TBA''), \text{ \&c.}$$

Hence $$\frac{(\mathrm{TBA}'')}{(\mathrm{TCA}'')}\cdot\frac{(\mathrm{TCB}'')}{(\mathrm{TAB})}\cdot\frac{(\mathrm{TAC}'')}{(\mathrm{TBC}'')}=1.$$

But $\frac{(\mathrm{TBA}'')}{(\mathrm{TCA}'')}$ is the ratio in which α divides BC, &c. Hence α, β, γ are collinear.

It is to be noticed that if ABC, A″B″C″ be two given triangles, and T a point such that TA″, TB″, TC″ cut BC, ... &c., in three collinear points, the locus of T is given by the equation

$$\frac{(\mathrm{TBA}'')}{(\mathrm{TCA}'')}\frac{(\mathrm{TCB}'')}{(\mathrm{TAB}'')}\frac{(\mathrm{TAC}'')}{(\mathrm{TBC}'')}=1.$$

This locus is a cubic, which degenerates into a conic through ABC and a straight line when A″, B″, C″ are in the same straint line; since A″B″C″ is then a part of the locus.

III. *Solution by the* PROPOSER.

Let the point be taken as origin, and the three sides be

$$ax+by+c=0,\quad a'x+b'y+c'=0,\quad a''x+b''y+c''=0;$$

then the equation of the conic will be of the form,

$$(a'x+\ldots)(a''x+\ldots)+(ax+\ldots)(a''x+\ldots)+(ax\ldots)(a'x\ldots)=0;$$

or $$Ax^2+Bxy+Cy^2+\ldots\ldots+\ldots\ldots=0,$$

where $$A=a'a''+aa''+aa',\quad C=b'b''+bb''+bb',$$
$$B=a'b''+a''b'+ab''+a''b+ab'+a'b,$$

and the absolute term $c'c''+cc''+cc'\equiv 0$, because the origin is on the curve. Write $d=a'b''-a''b'$, $d'=a''b-ab''$, $d''=ab'-a'b$, then the equation of the line drawn through the origin parallel to the diameter bisecting the line $ax+by+c=0$ will easily be found to be

$$\left\{(a+a')d'-(a+a'')d''\right\}x+\left\{(b+b')d'-(b+b'')d''\right\}y=0;$$

and the coordinates of the point of intersection of this line with $ax+by+c=0$ will be found to be

$$x=\left\{(b+b')d'-(b+b'')d''\right\}\frac{c}{2d'd''},\quad y=\left\{(a+a'')d''-(a+a')d'\right\}\frac{c}{2d'd''}.$$

Similarly, if (x', y'), (x'', y'') are the coordinates of the points in which the sides $a'x+\ldots=0$, $a''x+\ldots=0$ are met by lines drawn through the origin parallel to the diameters bisecting those sides respectively, we have

$$x'=\left\{(b'+b'')d''-(b'+b)d\right\}\frac{c'}{2dd''},\quad y'=\left\{(a'+a)d-(a'+a'')d''\right\}\frac{c'}{2dd''}$$

$$x''=\left\{(b''+b')d-(b''+b)d'\right\}\frac{c''}{2dd'},\quad y''=\left\{(a''+a)d'-(a''+a')d\right\}\frac{c''}{2dd}.$$

From the above values for (x, y), (x', y'), (x'', y'') it will follow that

$$xy'-x'y=-(d^2+d'^2+d''^2-2d'd''-2dd''-2dd')\frac{cc'}{4dd'd''}$$

and, by symmetry,

$$x''y-xy''=-(d^2+d'^2+d''^2-2d'd''-2dd''-2dd')\frac{cc}{4dd'd'},$$

$$x'y''-x''y'=-(d^2+d'^2+d''^2-2d'd''-2dd''-2dd')\frac{c'c''}{4dd'd'},$$

whence

$$xy'-x'y+x''y-xy''+x'y''-x''y'=$$

$$-\frac{d^2+d'^2+d''^2-2d'd''-2dd''-2dd'}{4dd'd''}(cc'+cc''+c'c'');$$

but since $cc'+cc''+c'c''=0$, as remarked before, on account of the origin being on the curve, we have the condition for the points (x, y), (x', y'), (x'', y'') being collinear, satisfied.

In the case of the ellipse the result is easily satisfied by projecting the figure into a circle orthogonally.

IV. *Solution by* STEPHEN WATSON.

Let ABC be any inscribed triangle; take $BC=a$, $BA=c$, as axes, then the equation of the conic is

$$y^2+Ax^2+Bxy-cy-aAx=0,$$

and the coordinates of its centre,

$$x_1=\frac{cB-2aA}{B^2-4A},\ y_1=\frac{A(aB-2c)}{B^2-4A}\ \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and if $y=mx$ be any line joining B to any point (x_2, y_2) in the conic

$$x_2=\frac{cm+aA}{m(m+B)+A},\ y_2=mx_2\ \ldots\ldots\ldots\ldots\ldots (2).$$

From the above, the equations of the lines through (2), parallel to those through (1), and bisecting BC, CA, AB respectively, are easily found to be

$$y-y_2=-\frac{2A}{B}(x-x_2)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

$$y-y_2=\frac{cB-2aA}{aB-2c}(x-x_2)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

$$y-y_2=-\frac{B}{2}(x-x_2)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

The line through the intersection of (3) and BC, and (5) and AB, is

$$\frac{Ax}{Bm+2A}+\frac{y}{2m+B}=\tfrac{1}{2}x_2\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6),$$

and the coordinates (x', y') of the intersection of (6) with the line BC, whose equation is $\left(\frac{x}{a}+\frac{y}{c}=1\right)$, are

$$x'=\frac{a(Bm+2A)}{2\{m(m+B)+A\}},\ y'=\frac{cm(2m+B)}{2\{m(m+B)+A\}},$$

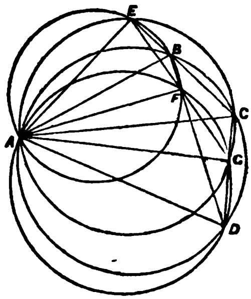
therefore $\frac{x'-x_2}{y'-y_2}=\frac{aB-2c}{cB-2aA}$; hence (x', y') fulfil (4), and the lines (3), (4), (5) meet in a point.

2444. (Proposed by A. RENSHAW.)—Upon any three chords (AB, AC, AD) of a circle as diameters circles are drawn, cutting one another again in E, F, G; prove that EFG is a straight line.

I. *Solution by* Professor HIRST.

This theorem is the inverse of the well known one that the circle circumscribed to the triangle formed by any three tangents to a parabola passes through the focus. To render this evident, it is merely necessary to take the point A as the centre of inversion. The given circle ABCD then becomes a straight line B′C′D′, and the three circles on AB, AC, AD as diameters become three right lines through B′, C′, D′, respectively, perpendicular to AB′, AC′, AD′, three tangents, therefore, to a parabola of which A is the focus and B′C′D′ the tangent at the vertex.

II. *Solution* by H. MURPHY; H. TOMLINSON; *the* PROPOSER; *and others.*



The angles AGD and AGC are both right angles, being those in a semicircle; therefore CGD is a straight line. Similarly CBE and BFD are straight lines. Hence CBD is a triangle inscribed in the original circle, and from a point A on its circumference perpendiculars AE, AF, AG are drawn to its sides; therefore, by a well-known property,* their three feet E, F, G are in the same straight line.

Otherwise: ∠ ADC = ABE = AFE, therefore AFE + AFG = ADG + AFG = 2 right angles; therefore E, F, G are in the same straight line.

[An analytical solution is given in Salmon's *Conics*, p. 99, ex. 7; and another in Salmon's *Higher Plane Curves*, p. 240, ex. 2.]

1959. (Proposed by W. S. BURNSIDE, M.A.)—If the roots of the equation $(a, b, c, d, e)(x, 1)^4 = 0$ are $\alpha, \beta, \gamma, \delta$, find the equation whose roots are $(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)$, &c.

Solution by MATTHEW COLLINS, B.A.; S. BILLS; *and others.*

We have
$$(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta) = 2\alpha^3-(\alpha+\beta+\gamma+\delta)\,\alpha^2 + (\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta+\alpha\beta\gamma)-2\beta\gamma\delta;$$
but as $\alpha, \beta, \gamma, \delta$ are the roots of the equation
$$ax^4+4bx^3+6cx^2+4dx+e=0 \ldots\ldots\ldots\ldots (1),$$
we have
$$\Sigma(\alpha) = -\frac{4b}{a}, \quad \Sigma(\beta\gamma\delta) = -\frac{4d}{a}, \quad \beta\gamma\delta = \frac{e}{a\alpha};$$

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* See Question 2395, on p. 44 of this volume of the *Reprint*.

hence $y = (\alpha-\beta)(\alpha-\gamma)(\alpha-\delta) = 2\alpha^3 + \frac{4b}{a}\alpha^2 - \frac{4d}{a} - \frac{2e}{a\alpha}$,

therefore, restoring x for α, and putting $4z$ for y, we have

$$ax^4 + 2bx^3 - 2(d+az)x - e = 0 \ldots\ldots\ldots\ldots (2).$$

(1) + (2) gives $ax^3 + 3bx^2 + 3cx + (d-az) = 0 \ldots\ldots\ldots\ldots (3);$

(1) − (2) gives $bx^3 + 3cx^2 + (3d+az)x + e = 0 \ldots\ldots\ldots\ldots (4).$

Now in Meyer Hirsch's *Algebra*, (pp. 119—121) it is shown that the result of eliminating x between the two equations

$$p + qx + rx^2 + sx^3 \text{ and } p' + q'x + r'x^2 + s'x^3 \text{ is}$$

$$(pq'-p'q)(sr'-s'r)(rq'-r'q) + 2(pq'-p'q)(sp'-s'p)(sr'-s'r)$$
$$+ (sp'-s'p)^3 + (pr'-p'r)^2(sr'-s'r)$$
$$+ (pr'-p'r)(sp'-s'p)(sq'-s'q) - (pq'-p'q)(sq'-s'q)^2 = 0 \ldots\ldots\ldots (5).$$

Substituting in (5) for p, p', &c., the coefficients in (3) and (4), it becomes

$$9(ac-b^2)(3d^2-3ce-2adz-a^2z^2)(3bd-3c^2+abz)$$
$$+6(ac-b^2)(3d^2-3ce-2adz-a^2z^2)(ae-bd+abz)$$
$$+(ae-bd+abz)^3 + 27(cd-be+acz)^2(ac-b^2)$$
$$+3(cd-be-acz)(ae-bd+abz)(3ad-3bc+a^2z)$$
$$+(3ce-3d^2+2adz+a^2z^2)(3ad-3bc+a^2z)^2 = 0,$$

or

$$a^6z^4 + 8a^3(a^2d-3abc+2b^3)z^3$$
$$+6a^2(a^2ce-3a^2d^2+14abcd-ab^2e-9ac^3-8b^3d+6b^2c^2)z^2$$
$$+\{81(d^2-ce)(bd-c^2)(ac-b^2)+18(d^2-ce)(ae-bd)(ac-b^2)+(ae-bd)^3$$
$$+9(cd-be)(ae-bd)(ad-bc)+27(ce-d^2)(ad-bc)^2\} = 0 \ldots (6).$$

[The coefficients in equation (6) may be readily reduced to the forms given in equation (F) of Mr. Burnside's solution. Moreover, equations (B) and (D) of that solution may be deduced from the foregoing as follows:—

Assume $x = v - \frac{b}{a}$; then equations (3) and (4) become

$$a^3v^3 + 3a(ac-b^2)v + 2b^3 - 3abc + a^2d - a^3z = 0 \ldots\ldots\ldots\ldots (3'),$$

$$a^3bv^3 + 3a^2(ac-b^2)v^2 + (3ab^3 - ba^2bc + 3a^3d + a^4z)v$$
$$-b^4 + 3ab^2c - 3a^2bd + a^3e - a^3bz = 0 \ldots\ldots (4').$$

(4′) − b (3′) gives $3a^2(ac-b^2)v^2 + a(2b^3 - 3abc + a^2d + a^3z)v$

$$-3b^4 + 6ab^2c - 4a^2bd + a^3e = 0 \ldots\ldots\ldots (5').$$

Now in Mr. Burnside's notation (3′) and (5′) may be written

$$a^3v^3 + 3u\mathrm{H}v + \mathrm{G} - a^3z = 0,\ 3a^2\mathrm{H}v^2 + a(a^3z + 3\mathrm{G}) + a^2\mathrm{I} - 3\mathrm{H}^2 = 0;$$

and they are thus seen to be *identical* with Mr. Burnside's equations.]

II. *Solution by the* PROPOSER.

Since the required equation has for its coefficients functions of the differences of $\alpha, \beta, \gamma, \delta$; the result will not lose its generality by removing the second term from $(a, b, c, d, e)(x, 1)^4 = 0$, When this is done it is found to be

$$a^4x^4+6a^2Hx^2+4aGx+a^2I-3H^2=0 \text{ .... (A)},$$

where $G=a^2d-3abc+2b^3,\quad H=ac-b^2,\quad I=ae-4bd+3c^2,$
and $J=ace+2bcd-ad^2-eb^2-c^3.$

Now it is easily seen that

$$a^3(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)=4(a^3\alpha^3+3aH\alpha+G)=4a^3z, \text{ suppose;}$$

and α must then be eliminated between the two equations

$$a^3\alpha^3+3aH\alpha+G-a^3z=0,\quad a^4\alpha^4+6a^2H\alpha^2+4aG\alpha+a^2I-3H^2=0 \text{ .. (B, C)}.$$

These are easily reduced to the two quadratics

$$3a^2H\alpha^2+a(a^3z+3G)\alpha+a^2I-3H^2=0 \text{ .... (D)},$$

$$a^2(3G+a^3z)\alpha^2+(a^3I-12aH^2)\alpha+3Ha^3z-3HG=0 \text{...... (E)};$$

and the result of the elimination of α between (D) and (E) is

$$a^6z^4+8a^3Gz^3-(18a^3J-12a^2IH)z^2+I^3-27J^2=0 \text{ ...... (F)}.$$

That the coefficient of z must vanish in (F) is otherwise evident from the fact that if the given equation have equal roots, that is if $I^3-27J^2=0$, two of the roots of the resulting equation must vanish.

2336. (Proposed by S. Roberts, M.A.)—Find the locus of points such that the chord of contact of tangents drawn to a given conic from any one of the points is of constant length.

Solution by the Rev. J. L. Kitchin, M.A.; W. Chadwick; *and others.*

Let (X, Y) be the coordinates of the point from which tangents are drawn; (x_1, y_1), $(x_2. y_2)$ the respective points of contact; then we have

$$\frac{Xx_1}{a^2}+\frac{Yy_1}{b^2}=1,\quad \frac{Xx_2}{a^2}+\frac{Yy_2}{b^2}=1;\ \therefore\ \frac{X}{a^2}(x_1-x_2)+\frac{Y}{b^2}(y_1-y_2)=0 \text{ ...(1)}.$$

Let ρ, which is of constant length, be the distance of these points from each other; then, if θ denote the angle which ρ makes with axis of x, we have

$$x_1-x_2=\rho\cos\theta,\quad y_1-y_2=\rho\sin\theta \text{ .... (2)};$$

therefore $$\frac{(x_2+o\cos\theta)^2}{a^2}+\frac{(y_2+\rho\sin\theta)^2}{b^2}=1,$$

therefore $$2\left(\frac{x_2\cos\theta}{a^2}+\frac{y_2\sin\theta}{b^2}\right)+\rho\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)=0 \text{ .. (3)}.$$

From (1) and (2) we find $\tan\theta=-\dfrac{b^2X}{a^2Y}$; hence substituting in (3) we get

$$2(x_2Y-y_2X)+\frac{(b^2X^2+a^2Y^2)\rho}{(b^4X^2+a^4Y^2)^{\frac{1}{2}}}=0 \text{ .... (4}$$

From (4) and $\dfrac{Xx_2}{a^2}+\dfrac{Yy_2}{b^2}=1$, we get

$$x_2 = \frac{a^2b^2X}{b^2X^2+a^2Y^2} - \frac{\rho a^2Y}{2(b^4X^2+a^4Y^2)^{\frac{1}{2}}},\quad y_2 = \frac{a^2b^2Y}{b^2X^2+a^2Y^2} + \frac{\rho b^2X}{2(b^4X^2+a^4Y^2)^{\frac{1}{2}}}.$$

Hence substituting in $\frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} = 1$, we get for the equation of the locus

$$\rho^2(b^2X^2+a^2Y^2)^2 = 4(b^4X^2+a^4Y^2)(b^2X^2+a^2Y^2-a^2b^2) \;\ldots\ldots\; (5),$$

a curve of the fourth degree. The corresponding locus for the hyperbola is obtained by putting $-b^2$ for b^2. The parabola is a very simple case, and is here omitted.

[*Otherwise*: if (h, k) be the *middle point* of a chord of constant length in the conic $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the *length* of this chord is given by the expression

$$\rho^2 = 4\left(\frac{b^4h^2+a^4k^2}{b^2h^2+a^2k^2}\right)\left(1 - \frac{b^2h^2+a^2k^2}{a^2b^2}\right) \;\ldots\ldots\ldots\ldots\ldots\; (A).$$

Moreover, we may write the *equation* of this line in the two following forms, (B) as a chord whose middle point is at the point (h, k), and (C) as the polar of the point (X, Y); viz.,

$$\frac{hx}{a^2} + \frac{ky}{b^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2}, \ldots\ldots (B),\quad \frac{Xx}{a^2} + \frac{Yy}{b^2} = 1 \;\ldots\ldots\ldots\; (C).$$

Comparing (B) with (C) we obtain

$$\frac{h}{X} = \frac{k}{Y} = \frac{b^2h^2+a^2k^2}{a^2b^2} = \left(\frac{b^4h^2+a^4k^2}{b^4X^2+a^4Y^2}\right)^{\frac{1}{2}} = \left(\frac{b^2h^2+a^2k^2}{b^2X^2+a^2Y^2}\right)^{\frac{1}{2}},$$

whence $$\frac{b^4h^2+a^4k^2}{b^2h^2+a^2k^2} = \frac{b^4X^2+a^4Y^2}{b^2X^2+a^2Y^2},\ \text{and}\ \frac{b^2h^2+a^2k^2}{a^2b^2} = \frac{a^2b^2}{b^2X^2+a^2Y^2} \;\ldots\; (D).$$

Substituting from (D) in (A) we obtain, for the required locus, the same equation (5) as in the foregoing solution.]

NOTE ON QUESTION 1990. BY PROFESSOR CAYLEY.

The theorem of paragraph 4,* (ascribed by Professor Sylvester to Mr. Crofton,) that "if a circle and a straight line be cut by any transversal in three points, these will be the foci of a system of Cartesian ovals having double contact with one another at two fixed points," may be enunciated under a more complete form, as follows:—

If in a given circle the chords PP_1, BC meet in A, then each of the two Cartesians, foci A, B, C, which pass through P, will also pass through P_1; and moreover, if α, α' be the diametrals of the chord PP_1 (that is, the extremities of the diameter at right angles to PP_1) then the tangents at P, P_1 to one of the Cartesians will be αP, αP_1 respectively, and to the other of them $\alpha' P$, $\alpha' P_1$ respectively, these tangents being thus independent of the position of the chord BC; and thence also thus;

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* See *Reprint*, Vol. VI., p. 88

Given the points A, B, C *in lineâ*, and the point P;
through P, B, C draw a circle (A) and let PA meet this in P_1,
" P, C, A " (B) " PB " P_2,
" P, A, B " (C) " PC " P_3,
then each of the Cartesians, foci A, B, C, which pass through P will also pass through P_1, P_2, P_3; and if
α, α' are the diametrals of PP_1 in circle (A),
β, β' " PP_2 " (B),
γ, γ' " PP_3 " (C),
then (the points of the several pairs being properly selected) the points (α, β, γ) and the points $(\alpha', \beta', \gamma')$ will each lie in a line through P, viz., the lines $P\alpha\beta\gamma$ and $P\alpha'\beta'\gamma'$ will be the tangents at P to the two Cartesians respectively.

The two Cartesians meet in the points P, P_1, P_2, P_3, and in the symmetrically situated points in regard to the axis ABC; the theorem contains as part of itself the well-known property that the two Cartesians cut at right angles at each of their points of intersection; it gives moreover the construction for the following problem:—given the foci A, B, C, and one intersection P of a pair of triconfocal Cartesians, to find the remaining intersections, and the tangents at each of the intersections.

AT RANDOM: No. VI. By W. S. B. Woolhouse, F.R.A.S.

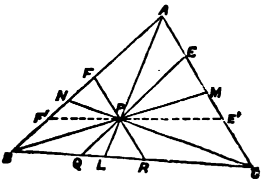
In an article contained in the August Number of the *Educational Times*, (*Reprint*, Vol. VIII., p. 35), Professor Whitworth seems to labour under some misapprehension as to an inaccuracy which has arisen in the statement of my Question (2420) proposed in the June Number. It is, in fact, a pure inadvertency, which I should not have failed to discover in drawing up a solution to the question, and has no reference whatever to the contorted interpretation of language which Professor Whitworth has incautiously chosen to assume. In the rough working of the question I was under the impression that the permutations of each combination were included, but it appears that only the combinations, inclusive of repetitions, are taken into account. The question therefore, instead of "Three numbers," &c., should have been stated "A set of three numbers," &c. It is, however, unnecessary to make any formal correction, as I perceive the substance of the question has been given by Professor Whitworth as the second part of his question (2434) in the July Number. I may, however, be permitted to observe that in his enunciation of this second part an inaccuracy occurs; the phrase "each one less than *n*" should be "each one not exceeding *n*." But I cannot bring my mind to attribute this oversight to looseness of language, notwithstanding Professor Whitworth has at the same time so pointedly reminded us that "by due care we can generally so express ourselves as to preclude any doubt about the meaning of our most unqualified statements."

Professor Whitworth further remarks, that "a precisely similar ambiguity occurs in Mr. Woolhouse's Question (2433) in the July Number." He asks, do I "mean that five points *are* selected at random, or that a selection of five points *is* made at random?" It would appear to me that in making this animadversion Professor Whitworth had not given the subject that attention which he might have done on assuming the office of a critic. Where the number of possible selections is infinite, he ought to have satisfied himself that it is virtually the same whether the five points are selected separately or as a group, and that the ambiguity he fancied was a myth of his own imagination.

Before concluding these few lines I may observe that Professor Whitworth and myself are in perfect agreement as to the general interpretation and usually defined signification of the phrase "at random;" and it is only right to add that I am obliged to Professor Whitworth for having suggested the inadvertency in the enunciation of Question (2420).

2359. (Proposed by EXHUMATUS.)—A point is taken at random in each side of a triangle ABC, so forming a second triangle DEF; determine the locus of those points within ABC which have a given probability p of being enclosed within the triangle DEF.

I. *Solution by* STEPHEN WATSON.



Let P be a point in the required locus, through which draw lines as in the diagram, and put respectively BL, LC, CM, &c. = a_1, a_2, b_1, &c., AL, AP, PL = l, l_1, l_2, and AE, AF = x, y. Then we easily find

$$LQ = \frac{a_2 l_2 x}{b l_1 - l x},\quad LR = \frac{a_1 l_2 y}{c l_1 - l y}.$$

Now P will lie within the triangle DEF when D is any point on QR; hence when E and F lie on AM and AN, the number of favourable positions of D, E, F, is

$$\int_0^{c_1}\int_0^{b_2}(LQ + LR)\,dy\,dx \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

When E and F lie on AM and BN, R lies beyond C, and the favourable positions are

$$\int_0^{b_2}(a_2 + LQ)\,c_2 dx \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Similarly, when E and F lie on CM and AN, the favourable positions are

$$\int_0^{c_1}(a_1 + LR)\,b_1 dy \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

When E and F lie on CM and BN. If E be at E, F may lie anywhere on F′N, and D on BC; hence the favourable positions are

$$\int_{b_2}^{b} (\mathrm{AF}'-c_1)\, adx = \int_{b_2}^{b} \left(\frac{a_2 c l_1 x}{alx - a_1 b l_1} - c_1\right) adx \ldots\ldots\ldots\ldots (4).$$

The above include the whole of the favourable cases, and the total number of positions is *abc*. Hence the chance of P lying within the triangle DEF, for every position of D, E, F is

$$\frac{(1)+(2)+(3)+(4)}{abc} = \frac{1}{abc}\left\{a_2 b_2 c_2 - a_2 b_1 c_1 - \frac{l_2}{l}(a_2 b_2 c + a_1 b c_1) + \frac{a_2 b_1 c l_1}{l}\right\}$$

$$+ \frac{l_1}{al^2}\left\{a_2 l_2 \log \frac{a_1 b l_1}{a_2 b_2 l_2} + a_1 l_2 \log \frac{a_2 c l_1}{a_1 c_1 l_2} + \frac{a_1 a_2 l_1}{a} \log \frac{bc}{b_2 c_1}\right\} = p \ldots (5).$$

Now (x, y, z) being the triangular coordinates of P, so that $x+y+z=1$, we have

$\frac{l_2}{l} = x, \frac{l_1}{l} = 1-x$; also $\frac{a_1}{a_2} = \frac{\Delta\,\mathrm{APB}}{\Delta\,\mathrm{APC}} = \frac{z}{y}$, $\therefore \frac{a_1}{a} = \frac{z}{1-x}$, and $\frac{a_2}{a} = \frac{y}{1-x}$;

with similar values for $\frac{b_1}{b}$, $\frac{b_2}{b}$, &c.; hence (5) reduces to

$$x(1-x)\log\left(\frac{1-x}{x}\right) + y(1-y)\log\left(\frac{1-y}{y}\right) + z(1-z)\log\left(\frac{1-z}{z}\right) = p,$$

or
$$\left(\frac{1-x}{x}\right)^{x(1-x)} \left(\frac{1-y}{y}\right)^{y(1-y)} \left(\frac{1-z}{z}\right)^{z(1-z)} = \epsilon^{p} \ldots\ldots\ldots (6),$$

which is the equation of the locus required.

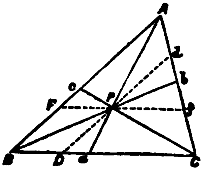
COR. 1. If AL, BM, CN be the internal bisectors of the angles A, B, C, then $\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = \frac{1}{2s}$; and (6) gives

$$p = \frac{a(b+c)}{4s^2}\log\left(\frac{b+c}{a}\right) + \frac{b(c+a)}{4s^2}\log\left(\frac{c+a}{b}\right) + \frac{c(a+b)}{4s^2}\log\left(\frac{a+b}{c}\right),$$

which is the chance that the point of the intersection of internal bisectors lies within the triangle DEF.

COR. 2. If AL, BM, CN bisect the sides, then $x = y = z = \frac{1}{3}$; and (6) gives $p = \frac{2}{3}\log 2$, which is the chance that the point of intersection of the bisectors of the sides lies within the triangle DEF.

II. *Solution by* MORGAN JENKINS, B.A.



Let P be a point within the triangle; *a*, *b*, *c* the points in which AP, BP, CP meet BC, CA, AB, respectively; D any point in B*a*; F any point in BA; *d*, *f* the points in which DP, FP meet AC. Then if F be in B*c*, the favourable cases will be when E falls within *df*; but if F be in *c*A, the favourable cases will be when E falls within C*d*.

Now, using triangular coordinates, the several points may be denoted as follows:—

$$(P)\ldots(x, y, z);\quad (D)\ldots(0, \rho, 1-\rho);\quad (F)\ldots(1-\tau, \tau, 0);$$

$$(d)\ldots\left(\frac{\rho x}{\rho-y}, 0, \frac{\rho(y+z)-y}{\rho-y}\right);\quad (f)\ldots\left(\frac{\tau(x+y)-y}{\tau-y}, 0, \frac{\tau z}{\tau-y}\right).$$

The chance of P being within DEF, if D be any point in B*a*, will be

$$\frac{\delta(\text{BD})}{\text{BC}} \cdot \frac{\delta(\text{BF})}{\text{BA}} \cdot \frac{df}{\text{AC}} + \frac{\delta(\text{BD})}{\text{BC}} \cdot \frac{\delta(\text{BF})}{\text{BA}} \cdot \frac{\text{C}d}{\text{AC}}.$$

Now BD = $(1-\rho)$ BC, and BF = $(1-\tau)$ BA; hence the sum of these chances, when D lies anywhere in B*a*, is

$$\int_{\frac{y}{y+z}}^{1}\int_{\frac{y}{x+y}}^{1}\left(\frac{\rho x}{\rho-y} - \frac{\tau(x+y)-y}{\tau-y}\right) d\rho d\tau + \int_{\frac{y}{y+z}}^{1}\int_{0}^{\frac{y}{x+y}} \frac{\rho x}{\rho-y}\, d\rho d\tau$$

$$= \int_{\frac{y}{y+z}}^{1} \frac{\rho x}{\rho-y}\, d\rho - \frac{z}{y+z}\int_{\frac{y}{x+y}}^{1} \frac{\tau(x+y)-y}{\tau-y}\, d\tau$$

$$= xy \log\frac{(x+z)(y+z)}{xy} + \frac{yz^2}{y+z}\log\frac{(x+y)(x+z)}{yz}.$$

It will be found that the chance when D lies in *a*C may be obtained from the foregoing expression by interchanging y and z; hence we have

$$p = xy \log\frac{(x+z)(y+z)}{xy} + xz\log\frac{(x+y)(y+z)}{xz}$$

$$+ \frac{yz^2+y^2z}{y+z}\log\frac{(x+z)(x+y)}{yz}$$

$$= x(1-x)\log\left(\frac{1-x}{x}\right) + y(1-y)\log\left(\frac{1-y}{y}\right) + z(1-z)\log\left(\frac{1-z}{z}\right);$$

therefore $\left(\frac{1-x}{x}\right)^{x(1-x)} \left(\frac{1-y}{y}\right)^{y(1-y)} \left(\frac{1-z}{z}\right)^{z(1-z)} = \epsilon^{p}$

is the equation, in triangular coordinates, of the required locus.

2397. (Proposed by Professor WHITWORTH.) — If a body of homogeneous fluid be thoroughly shaken and allowed to come to rest again, the chance that no particle of the fluid now occupies its original position is ϵ^{-1}, where ϵ is the base of Napierian logarithms.*

I. *Solution by the* PROPOSER; *and* C. M. INGLEBY, LL.D.

First suppose there were a finite number n of particles in the fluid; and let them be distinguished severally by the letters $\alpha, \beta, \gamma \ldots \kappa$. Let N denote the number of ways of arranging the n particles; also let (A) express the condition that α is in its original place, and (a) the condition that α is out of its original place. Let (B) and (b) denote the corresponding conditions for β, and so on.

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* For the history of this problem see Todhunter's *History of the Theory of Probability*, Arts. 160—162, 196, 204, 430, 627, &c.

Then we have $\mathrm{N} = \Pi(n)$; but $\mathrm{N}(\mathrm{A}) = \Pi(n-1)$; hence by subtraction $\mathrm{N}(a) = \Pi(n) - \Pi(n-1)$. But the condition (B) would leave us $(n-1)$ things instead of n to deal with. Hence writing $(n-1)$ for n in the last equation, we have $\mathrm{N}(a\mathrm{B}) = \Pi(n-1) - \Pi(n-2)$.

Therefore, subtracting, $\mathrm{N}(ab) = \Pi(n) - 2\Pi(n-1) + \Pi(n-2)$.

Similarly, writing $n-1$ for n in this equation, we get

$$\mathrm{N}(ab\mathrm{C}) = \Pi(n-1) - 2\Pi(n-2) + \Pi(n-3),$$

and subtracting, $\mathrm{N}(abc) = \Pi(n) - 3\Pi(n-1) + 3\Pi(n-2) - \Pi(n-3)$,

and so on. Hence we obtain

$$\mathrm{N}(abc\ldots k) = \Pi(n) - \frac{n}{1}\Pi(n-1) + \frac{n(n-1)}{1.2}\Pi(n-2) - \text{\&c. to } (n+1) \text{ terms.}$$

$$= \Pi(n)\left\{1 - 1 + \frac{1}{2} - \frac{1}{2.3} + \ldots\ldots \text{to } (n+1) \text{ terms}\right\}.$$

Therefore the chance that *all* the n particles are misplaced is

$$1 - 1 + \frac{1}{2} - \frac{1}{2.3} + \ldots\ldots \text{to } (n+1) \text{ terms},$$

which, when n becomes infinite, becomes the expansion of e^{-1}.

II. *Solution by* W. S. B. WOOLHOUSE, F.R.A.S.

1. The number of particles of the fluid is here assumed to be infinite, or indefinitely great. To determine the question generally, suppose the number to be finite. Consider a primary group of n things or symbols to be taken in a specified order, and on the $\Pi(n)$ permutations of the group being compared with it, let $\phi(n)$ denote the number of ways in which the n symbols are wholly displaced. Then it is evident that the number of ways having 1 symbol in its original place and $n-1$ symbols wholly displaced is equal to $n\phi(n-1)$; the number of ways having 2 symbols in their original places and the remaining $n-2$ symbols wholly displaced is equal to $\frac{1}{2}n(n-1)\,\phi(n-2)$; and so on. The total number is that of the permutations. That is,

$$\phi(n) + n\phi(n-1) + \tfrac{1}{2}n(n-1)\,\phi(n-2) \ldots\ldots + 1 = \Pi(n).$$

Similarly are obtained the equalities

$$\begin{aligned} \phi(n-1) + (n-1)\,\phi(n-2) \ldots\ldots + 1 &= \Pi(n-1) \\ \phi(n-2) \ldots\ldots + 1 &= \Pi(n-2) \\ \text{\&c.} \qquad & \quad \text{\&c.} \end{aligned}$$

Hence, by eliminating the depressed functions $\phi(n-1)$, $\phi(n-2)$, &c., from these relations, we get

$$\phi(n) = \Pi(n) - n\Pi(n-1) + n\frac{n-1}{2}\Pi(n-2) - n\frac{n-1}{2}\cdot\frac{n-2}{3}\Pi(n-3) + \text{\&c.}$$

$$= \Pi(n)\left(1 - 1 + \frac{1}{2} - \frac{1}{2.3} + \frac{1}{2.3.4} \text{ \&c. to } n+1 \text{ terms}\right)$$

$$= \Pi(n)\left(\frac{1}{2} - \frac{1}{2.3} + \frac{1}{2.3.4} \text{ \&c. to } n-1 \text{ terms}\right).$$

Therefore, if one of the $\Pi(n)$ permutations of the group be taken indiscriminately and compared with the primary arrangement, the chance that all the symbols are severally displaced is

$$p = \frac{1}{2} - \frac{1}{2.3} + \frac{1}{2.3.4} - \text{\&c. to } n-1 \text{ terms.}$$

According to the established formula

$$\epsilon^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{2.3} + \frac{x^4}{2.3.4} - \text{\&c.}$$

we have $\epsilon^{-1} = 1 - 1 + \frac{1}{2} - \frac{1}{2.3} + \frac{1}{2.3.4} - \text{\&c.}$

Thus it appears that the chance of the n symbols being wholly displaced is precisely equal to the expansion of ϵ^{-1} to $n+1$ terms, which is a remarkable result. It is further remarkable that, as the number n is consecutively increased, the value of the probability is alternately increased and diminished. In fact, it oscillates on both sides at the same time that it approximates nearer and nearer to its ultimate value ϵ^{-1}.

2. *Otherwise*: as there are $n-1$ positions of displacement for any one symbol, the probability of displacement for each symbol is $\frac{n-1}{n}$, and the probability that all the n symbols shall be displaced is

$$p = \left(\frac{n-1}{n}\right)^n = \left(1 - \frac{1}{n}\right)^n$$

$$= 1 - n \cdot \frac{1}{n} + \frac{n(n-1)}{2} \cdot \frac{1}{n^2} - \frac{n(n-1)(n-2)}{2.3} \cdot \frac{1}{n^3} + \text{\&c.}$$

And when n is increased indefinitely, this becomes

$$p = 1 - 1 + \frac{1}{2} - \frac{1}{2.3} - \text{\&c.} = \epsilon^{-1},$$

the same as before.

Or if $\frac{1}{n} = m$; then $p = (1-m)^{\frac{1}{m}}$, and

$$\log p = \frac{1}{m} \log(1-m) = \frac{1}{m} \int \frac{-dm}{1-m}$$

$$= \frac{1}{m} \int -dm\,(1 + m + m^2 + \text{\&c.})$$

$$= -\left(1 + \frac{m}{2} + \frac{m^2}{3} + \text{\&c.}\right)$$

$$= -\left(1 + \frac{1}{2n} + \frac{1}{3n^2} + \text{\&c.}\right)$$

$$\therefore p = \varepsilon^{-\left(1 + \frac{1}{2n} + \frac{1}{3n^2} + \text{\&c.}\right)}$$

And when $n = \infty$, $p = \epsilon^{-1}$.

NOTE.—It should be stated, that the methods of solution last given are not quite strict. The probability of displacement of any specified symbol is undoubtedly $\frac{n-1}{n}$, but the probability of the displacement of all the symbols, when n is a finite number, is not accurately equal to $\left(\frac{n-1}{n}\right)^n$, since these several simultaneous displacements are not independent. For it is obvious that the displacement of any one symbol must necessitate the displacement of the symbol which appertains to the place it occupies. The inaccuracy thus involved diminishes as the number n is increased, and becomes zero when n is infinite. But unless this is proved to be so, the solution is not conclusive even for this particular case. It will however be unnecessary here to adduce any such proof as the solution first given is perfectly general and accurate throughout.

2426. (Proposed by C. TAYLOR, M.A.) — If F_n denote the fraction $\frac{1+a^2+a^4\ldots\ldots+a^{2n}}{a+a^3+\ldots+a^{2n-1}}$, prove that, in general, if n is not $= 1$,

(1)...$F_n > \frac{n+1}{n} + \left(\sqrt{a}-\frac{1}{\sqrt{a}}\right)^2$, (2)...$F_n > \frac{1}{2}\,\frac{n+1}{n}\left(a+\frac{1}{a}\right)$, (3)...$\frac{n}{n+1}F_n$ increases with n.

Solution by C. M. INGLEBY, LL.D.

1. Summing the series, for numerator and denominator, in F_n, and converting the resulting fraction into a continued fraction, as in my solution of Quest. 1521, (*Reprint*, Vol. V., p. 99,) we obtain the expression,

$$F_n = a + \frac{1}{a} - \frac{1}{F_{n-1}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$= \left(\sqrt{a}-\frac{1}{\sqrt{a}}\right)^2 + 2 - \frac{1}{F_{n-1}} = \frac{n+1}{n} + \frac{n-1}{n} + \left(\sqrt{a}-\frac{1}{\sqrt{a}}\right)^2 - \frac{1}{F_{n-}}$$

$$= \frac{n+1}{n} + \left(\sqrt{a}-\frac{1}{\sqrt{a}}\right)^2 + \left(\frac{n-1}{n} - \frac{1}{F_{n-1}}\right) \ldots\ldots\ldots\ldots (2).$$

But, by Quest. 1521, $F_{n-1} > \frac{n}{n-1}$;

$\therefore\ \frac{n-1}{n} - \frac{1}{F_{n-1}}$ is positive; $\therefore$ from (2), $F_n > \frac{n+1}{n} + \left(\sqrt{a}-\frac{1}{\sqrt{a}}\right)^2$.

2. Since $F_{n-1} > \frac{n}{n-1}$ and $a + \frac{1}{a} > 2$;

$$\therefore\ F_{n-1} > \frac{n}{n-1}\cdot\frac{2}{a+\frac{1}{a}},\quad \therefore\ \frac{1}{F_{n-1}} < \frac{n-1}{2n}\left(a+\frac{1}{a}\right);$$

substituting, then, for $\frac{1}{F_{n-1}}$ in (1) the quantity $\frac{n-1}{2n}\left(a+\frac{1}{a}\right)$, the equation (1) becomes an inequality, viz.,

$$F_n > a+\frac{1}{a}-\frac{n-1}{2n}\left(a+\frac{1}{a}\right), \therefore F_n > \frac{1}{2}\frac{n+1}{n}\left(a+\frac{1}{a}\right).$$

3. This is a corollary from (2). Of course, if $\frac{n}{n+1}F_n$ increases with n, it is $> \frac{n-1}{n}F_{n-1}$ and this $< \frac{n-2}{n-1}F_{n-2}$, and so forth; till at last we have $\frac{2}{3}\frac{1+a^2+a^4}{a+a^3} > \frac{1}{2}\left(a+\frac{1}{a}\right)$; so that $\frac{n}{n+1}F_n > \frac{1}{2}\left(a+\frac{1}{a}\right)$. Conversely, if $\frac{n}{n+1}F_n$ do not increase with n, and we exclude equations, (as we must, since we should have at last a general equation between a function of n and a quantity that does not involve it), we have

$$\frac{n}{n+1}F_n < \frac{1}{2}\left(a+\frac{1}{a}\right), \text{ or } F_n < \frac{1}{2}\frac{n+1}{n}\left(a+\frac{1}{a}\right),$$

which is absurd, since the converse is established by Art. 2.

2447. (Proposed by Professor SYLVESTER.)—If in the development of $\sin^{-1}.x$, viz.,

$$\frac{x^1}{1}+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1.3}{2.4}\cdot\frac{x^5}{5}+\ldots\ldots$$

every number which occurs is increased by unity, prove that the result, *i.e.*,

$$\frac{x^2}{2}+\frac{2}{3}\cdot\frac{x^4}{4}+\frac{2.4}{3.5}\cdot\frac{x^6}{6}+\ldots\ldots$$

is equal to $\frac{1}{2}(\sin^{-1}x)^2$.

I. *Solution by* THOMAS SAVAGE, M.A.

Since $(\sin^{-1}x)^2$ is unaffected by a change in the sign of x, and vanishes with x, we may assume that

$$u = \tfrac{1}{2}(\sin^{-1}x)^2 = A_2\frac{x^2}{2}+A_4\frac{x^4}{4}+A_6\frac{x_6}{6}+\ldots.$$

$$\text{Now}\int u\,dx = \tfrac{1}{2}x(\sin^{-1}x)^2+(1-x^2)^{\frac{1}{2}}\sin^{-1}x-x = xu+(1-x^2)\frac{du}{dx}-x,$$

$$\therefore A_2\frac{x^3}{2.3}+A_4\frac{x^5}{4.5}+\ldots = A_2\frac{x^3}{2}+A_4\frac{x^5}{4}+\ldots+(1-x^2)(A_2x+A_4x^3+\ldots)-x,$$

whence, by equating the coefficients of corresponding powers of x, we have

$$A_2 = 1 \text{ and } \frac{A_{2n}}{2n\,(2n+1)} = A_{2n+2} - A_{2n} + \frac{A_{2n}}{2n} \text{ or } A_{2n+2} = \frac{2n}{2n+1}\,.A_{2n}.$$

Thus we obtain $A_4 = \frac{2}{3}$, $A_6 = \frac{2\,.\,4}{3\,.\,5}$, $A_8 = \frac{2\,.\,4\,.\,6}{3\,.\,5\,.\,7}$, &c.

Therefore $\frac{1}{2}(\sin^{-1}x)^2 = \frac{x^2}{2} + \frac{2}{3}\frac{x^4}{4} + \frac{2\,.\,4}{3\,.\,5}\cdot\frac{x^6}{6} + \ldots .$

II. *Solution by* M. Jenkins, B.A.; H. Tomlinson; *and others.*

We have $\frac{d}{dx}\left\{\frac{1}{2}(\sin^{-1}x)^2\right\} = (\sin^{-1}x)\,(1-x^2)^{-\frac{1}{2}}$

and $\frac{d^2}{dx^2}\left\{(\frac{1}{2}\sin^{-1}x)^2\right\} = (1-x^2)^{-1} + x\,(\sin^{-1}x)\,(1-x^2)^{-\frac{3}{2}}$

$$= (1-x^2)^{-1}\left\{1 + x\frac{d}{dx}\,.\,\frac{1}{2}(\sin^{-1}x)^2\right\} \ldots\ldots\ldots (1).$$

Assume that $\frac{d}{dx}\left\{(\frac{1}{2}\sin^{-1}x)^2\right\} = \sum_{r=0}^{r=\infty} A_{2r+1}\,x^{2r+1}$ where $A_1 = 1$ .. (2),

since it is evident that the series cannot contain *even* powers of x; then

$$\frac{d^2}{dx^2}\left\{(\tfrac{1}{2}\sin^{-1}x)^2\right\} = \sum_{r=0}^{r=\infty}(2r+1)\,A_{2r+1}\,x^{2r};$$

therefore from (1) and (2) we obtain

$$1 + \sum_{r=0}^{r=\infty}\left\{(2r+3)\,A_{2r+3} - (2r+1)\,A_{2r+1}\right\}x^{2r+2}$$

$$= 1 + \sum_{r=0}^{r=\infty} A_{2r+1}\,x^{2r+2}.$$

Hence $A_{2r+3} = \frac{2r+2}{2r+3}\,A_{2r+1}$; and therefore, since $A_1 = 1$, we have

$$A_3 = \frac{2}{3}, \text{ and } A_{2r+1} = \frac{2\,.\,4 \ldots 2r}{3\,.\,5 \ldots (2r+1)},$$

Hence by integration of (2) we obtain

$$\tfrac{1}{2}(\sin^{-1}x)^2 = \tfrac{1}{2}x^2 + \sum_{1}^{\infty}\frac{2\,.\,4 \ldots 2r}{3\,.\,5 \ldots (2r+1)}\cdot\frac{x^{2r+2}}{2r+2},$$

which proves the theorem.

[Another investigation is given in Todhunter's *Differential Calculus*, Art. 121 (4th ed.) For a more general form of the property, see Question 2481.]

2452. (Proposed by Professor SYLVESTER.)—Prove that if P, Q be two points chosen at random inside a tetrahedron ABCD, the chance is an even one of ABCPQ forming the angles of a *convex* solid.

I. *Solution by the* PROPOSER.

If we divide the whole of the tetrahedron ABCD into partial tetrahedra of which P is the vertex and each face in turn the base, it is clear that Q will lie in one of these. In like manner P will lie in one of the tetrahedra having Q for its vertex and one of the faces of ABCD for its base. Hence the sum of the chances of PQABC, PQABD, PQACD, PQBCD forming reentrant solids is *twice certainty,* and consequently the chance of PQABC being reentrant when ABCD is a regular tetrahedron is $\frac{1}{2}$, and the principle of hyperperspective invariable association leads to the belief that the conclusion remains good whatever the form of the limiting tetrahedron.

So in like manner we see instantaneously that the chance is $\frac{2}{3}$ of P, Q taken at random inside a triangle ABC forming with A, B the angles of a reentrant quadrilateral, at all events when ABC is equilateral, and then by perspective whatever the form of ABC.

NOTE.—The method of invariable association above suggested is open to very serious and possibly irremovable objections in point of legitimacy, whether for the triangle or the tetrahedron. The result itself is of course correct; in fact the method used for the triangle in the solution of Question 2371 (*Reprint,* Vol. VIII., p. 36) gives for the chance of non-convexity in the case of the tetrahedron

$$2\int_0^h dx\,.\,x^2(h-x) \div \int_0^h dx\,.\,x^2h, \text{ which is } \tfrac{1}{2}.$$

II. *Solution by* PROFESSOR EVERETT.

If the solid ABCPQ is not convex, either P is within ABCQ or Q within ABCP. These two events are mutually exclusive and equally probable, therefore the chance of ABCPQ being not convex is double the chance of Q being within ABCP.

Let h and $h-x$ denote the distances of D and P respectively from the base ABC. For each value of x the chance of Q being within ABCP is $\frac{h-x}{h}$; but the probabilities of different values of x are as x^2; hence the chance of Q being within ABCP is

$$\int_0^h dx\,\frac{h-x}{h}x^2 \div \int_0^h dx\,.\,x^2 = \frac{\frac{1}{3}-\frac{1}{4}}{\frac{1}{3}} = \tfrac{1}{4},$$

and the chance of ABCPQ being not convex is double of this, or $\frac{1}{2}$.

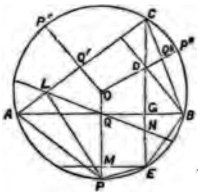
If instead of a tetrahedron ABCD the limiting solid be a prism on ABC as base, the chance of ABCPQ being not convex is

$$2\int_0^h dx\,\frac{h-x}{3h} \div \int_0^h dx = \frac{2}{3}\left(1-\frac{1}{2}\right) = \frac{1}{3}.$$

The former solution holds good for any pyramid or cone, and the latter for any prism or cylinder.

2405. (Proposed by H. Murphy.)—If from the middles of the arcs which subtend the three sides of an inscribed triangle, three sets of perpendiculars be drawn to the three sides of the triangle, and their feet joined, a new triangle will be formed; prove that the sum of the segments of the perpendiculars of the original triangle, between their point of intersection and the sides of the new triangle, is equal to the excess of the diameter of the circumscribing circle above the radius of the inscribed circle.

Solution by the Proposer; J. Dale; *and others.*



From P, the middle of the arc AB, draw the perpendiculars PL and PQ; then the line LQ gives the direction of one of the sides of the triangle referred to in the Question. Now, (Euc. I., 26,) DG = GE = QM; but in the triangles NQG and PME, the angle NQG = LQA = LPA, which is the complement of the angle PAL, and MEP is the complement of the same angle; therefore, (Euc. I., 26,) PM = NG; hence PQ = ND. In the same way P′Q′ and P″Q″ may be proved equal to the other two segments of the perpendiculars; but

$$\begin{aligned}PQ + P'Q' + P''Q'' &= R\left\{3-(\cos A+\cos B+\cos C)\right\}\\ &= 2R - R(\cos A+\cos B+\cos C-1)\\ &= 2R - r.\end{aligned}$$

Cor.—If P were taken anywhere in the arc AB, we should have, similarly, PQ = DN, which is the solution of an Honour Question, proposed in 1866, in Trinity College, Dublin.

2460. (Proposed by T. Cotterill, M.A.)—Find the conic which is the locus of a point such that the lines joining it to a pair of conjugate foci are harmonicals to the conic. The circular points at infinity being considered as conjugate foci, the three conics thus obtained and the conic in question have four points in common.

Solution by Professor Hirst.

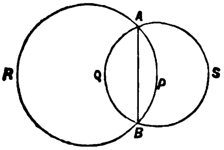
Let A, A′; B, B′; C, C′ be the three pairs of opposite intersections of any quadrilateral circumscribed to the given conic (Σ). Through A draw any right line x and connect its pole with A′ by a line x'. The pole of x', of course, lies in x; and x and x' are harmonicals relative to (Σ). But they are also corresponding rays of two homographic pencils (for to each ray x corresponds but one ray x' and *vice versâ*), hence the locus of their intersection x is a conic (X) which touches in A and A′ the connectors of these points with the pole of AA′. Moreover since, when either of the lines x or x'

touches the conic (Σ) the other passes through its point of contact, it is obvious that (X) passes through the four points of contact of the circumscribed quadrilateral. The same is true of the conic (Y) [or (Z)] which is the locus of the intersection of harmonicals passing through B and B′ [or C and C′]. Hence the three conics (X), (Y), (Z) pass through the same four points of (Σ).

When A and A′ coincide with the circular points at infinity we have obviously the theorem in the question.

NOTE WITH REFERENCE TO QUESTION 1829 (*Reprint*, Vol. IV., p. 77.)

BY SAMUEL ROBERTS, M.A.



It may be worth while to notice that if in the first case of a circle, PARB be the given circle, and with centre P thereon and radius $2k$ we draw a circle SAQB meeting the given circle in A and B; then, joining AB, the probability sought in the question is expressed by

$$\frac{2\,\text{Area AQB} + \text{Area APB}}{\text{Area PARB}}.$$

The corresponding expression for a sphere also gives the probability in that case: that is to say, we have for the sought probability

$$\frac{2\,\text{Vol. AQB} + \text{Vol. APB}}{\text{Vol. PARB.}},$$

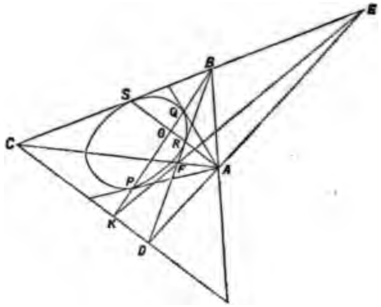
supposing the figure to be turned about RP.

I have not been able to obtain a satisfactory explanation of this result which ought to enable us to solve the problems without integration by taking a fixed point on the surface of the sphere or the circumference of the circle.

We seem however to get a clue to the matter by considering that if AQB be filled up with concentric arcs and APB with arcs touching at P and having their centres on RP, an element of an arc to radius ρ in AQB will subtend at P twice the angle which a corresponding element of an arc to radius ρ in APB will subtend at the same point. This would seem to imply that the elements of AQB must count twice when we seek the probability that a point taken at random in the circle PARB shall be not further than $2k$ from P.

2343. (Proposed by W. K. CLIFFORD, B.A.)—A is any point within or without a conic, B any point on its polar, CD a fixed straight line. Tangents BC, BD are drawn cutting CD in C, D. AD, AC meet BC, BD in E, F; show that EF is a fixed straight line and meets CD on the polar of A.

1. *Solution by the* **Rev. J. L. Kitchin, M.A.**



Let the conic, for convenience, be an ellipse, and A *without* the curve. Then it is plain that AOB is a triangle self conjugate with respect to the conic.

Taking this as the triangle of reference, we may write the equation to the conic

$l^2\alpha^2 + m^2\beta^2 - n^2\gamma^2 = 0$, or
$m^2\beta^2 = (n\gamma + l\alpha)(n\gamma - l\alpha) \dots (1)$;
therefore $n\gamma + l\alpha = 0 \quad \dots\dots (2)$
and $n\gamma - l\alpha = 0, \quad \dots\dots\dots\dots (3)$
are the equations of tangents from B, viz , BSC, BRD.

Assumed the equation to CD

$$p\alpha + q\beta + r\gamma = 0 \quad \dots\dots\dots\dots\dots\dots (4);$$

then since AD passes through the intersection of (3) and (4), and also through the point ($\beta = 0, \gamma = 0$), we get for its equation

$$lq\beta + (pn + rl)\gamma = 0 \dots\dots\dots\dots\dots (5).$$

Similarly the equation of AC is

$$lq\beta - (pn - rl)\gamma = 0 \quad \dots\dots\dots\dots (6).$$

Now EF passes through (2) and (5), and also through (3) and (6); hence its equation is $\quad -p\alpha + q\beta + r\gamma = 0 \quad \dots\dots\dots\dots\dots (7)$, which shows that EF is a *fixed* line: also it evidently passes through the intersection of $\alpha = 0$ and (4), or EFK is a straight line.

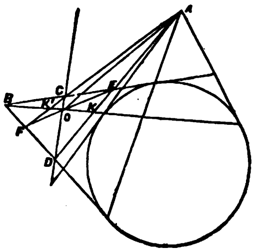
Obviously we might draw a figure, and the lines corresponding to those in the above diagram, either with B or O to start from; and the same conclusion would follow in each case. Hence the point may be either within or without the conic. Again, the only use made of the ellipse is to make the figure simpler; hence what is proved is true of any conic section.

It is easy to write the remaining two equations for the lines from B and O; viz., these are

$$p\alpha - q\beta + r\gamma = 0 \dots (8), \text{ and } p\alpha + q\beta - r\gamma = 0 \dots (9).$$

Of these, (8) passes through the intersection of CD and $\beta = 0$, or the polar of B; and (9) through the intersection of CD and $\gamma = 0$, or the polar of O.

II. *Solution by* **J. Dale**; **H. Tomlinson**; *and others.*



Let the polar of A meet the given line CD, and the lines AD, AC in O, K, K′ respectively, and join OA. Then the range (AKED) = (AK′FC), since both are harmonic; also OA is common to both ranges; moreover, (OK, OK′) and (OD, OC) are coincident; therefore OE and OF′ are in the same straight line; and since KK′, CD, and OA are constant in position, EF is also constant in position.

The same may be shown analytically, by referring the conic to any self-conjugate triangle, A and its polar being represented by ($y = 0$, $z = 0$) and $x = 0$, the equation being

$$L^2x^2 + M^2y^2 + N^2z^2 = 0.$$

The tangents to the conic from any point B on the polar of A will be

$$\mathrm{L}\,(\mathrm{M}^2y'^2+\mathrm{N}^2z'^2)^{\frac{1}{2}}\,x+\mathrm{MN}z'y+\mathrm{MN}y'z=0,$$
$$-\mathrm{L}\,(\mathrm{M}^2y'^2+\mathrm{N}^2z'^2)^{\frac{1}{2}}\,x+\mathrm{MN}z'y+\mathrm{MN}y'z=0.$$

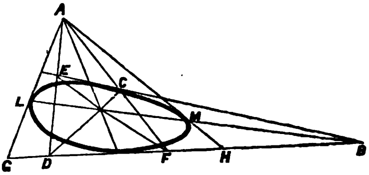
Let the equation to CD be $lx+my+nz=0$; then the equations to AED AFC are respectively

$$\mathrm{L}\,(\mathrm{M}y'^2+\mathrm{N}z'^2)^{\frac{1}{2}}\,(my+nz)+l\mathrm{MN}\,(z'y+y'z)=0,$$
$$\mathrm{L}\,(\mathrm{M}y'^2+\mathrm{N}z'^2)^{\frac{1}{2}}\,(my+nz)+l\mathrm{MN}\,(z'y+y'z)=0;$$

and thence we find the equation to EF to be $-lx+my+nz=0$, which represents a fixed straight line passing through the intersection of the line $lx+my+nz=0$ and $x=0$, that is, through the intersection of CD and the polar of A.

III. *Solution by the* Rev. R. H. Wright, M.A.

Taking AGH as the triangle of reference, where GH, HA, AG are respectively represented by $\alpha=0$, $\beta=0$, $\gamma=0$, the equations of the several lines may be expressed as follows:—



Conic........................$(l\alpha)^{\frac{1}{2}}+(m\beta)^{\frac{1}{2}}+(n\gamma)^{\frac{1}{2}}=0$;

Polar LM.....................$-l\alpha+m\beta+n\gamma=0$;

Tangent BC...................$l\alpha-2m\beta+2n\gamma=0$;

AD....$\beta-k\gamma=0$; AF....$\beta-h\gamma=0$;

EF....$l(h-k)\,\alpha+2\,(n+mk)\,\beta-2h\,(n+mk)\,\gamma=0$;

DC....$l(k-h)\,\alpha+2\,(n+mh)\,\beta-2k\,(n+mh)\,\gamma=0$;

By subtracting the two last equations, we have for the intersection of CD and EF,

$$-l\alpha+m\beta+n\gamma=0,$$

which is the polar of A; therefore the property is established. In the same manner we may obtain the reciprocal property.

Cor. 1.—From this it is manifest that if a conic be inscribed in a triangle; a tangent at the point where the line which joins the vertex with the point of contact of the opposite side cuts the conic, the line which joins the points of contact with the other two sides, and the base of the triangle, are concurrent.

Cor. 2.—The locus of intersection of EF and CD will be a straight line, the polar of A.

IV. *Solution by* W. H. Laverty.

Project the conic into a circle, and A into its centre, then the properties being entirely about position, will hold good in the general case, if they do in the particular one. But B will be somewhere on the line at infinity, and BEC, BFD will be parallel tangents. Also EF is evidently parallel to CD, and at a fixed distance from it, that is, EF is a fixed right line. And it meets CD on the line at infinity, which is the polar of A.

280. (Proposed by M. W. CROFTON, B.A.)—A Cartesian oval or ellipse
ut by an *axial* circle [*i.e.* a circle bisected by the axis] in two points
ose vectorial coordinates referred to two of the foci are (r, r') and (s, s').
ow that if a circle concentric to the former be drawn *touching* the curve,
e vectorial coordinates of either point of contact are $\left\{\frac{1}{2}(r+s),\ \frac{1}{2}(r'+s')\right\}$.

I. *Solution by* JAMES DALE.

Taking one of the foci as origin, and putting c for the distance between the foci, the polar equation of a Cartesian oval is

$$lr + m\,(r^2-2cr\cos\theta+c^2)^{\frac{1}{2}} = c \text{ ....(1).}$$

The equation of an axial circle whose radius is k, and whose centre is at a distance d from the origin, is

$$r^2-2dr\cos\theta+d^2-k^2=0 \text{ ....(2).}$$

Eliminating $\cos\theta$ from (1) and (2) gives

$$\left\{l^2d+m^2(c-d)\right\}r^2-2lcdr+\text{const.}=0 \text{ ....(3);}$$

therefore if r and s be the radii common to the circle and curve, we have

$$\tfrac{1}{2}(r+s)=\frac{lcd}{l^2d+m^2(c-d)},\text{ and similarly } \tfrac{1}{2}(r'+s')=\frac{mc(c-d)}{l^2d+m^2(c-d)}.$$

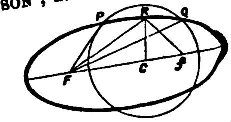
If (2) be taken to represent a tangential circle, then the roots of (3) must be equal; so that, if R be the coordinate of the point of contact, we have

$$\text{R}=\frac{lcd}{l^2d+m^2(c-d)},\text{ and S}=\frac{mc(c-d)}{l^2d+m^2(c-d)};$$

therefore $\text{R}=\frac{1}{2}(r+s),\ \text{S}=\frac{1}{2}(r'+s').$

In the ellipse $l=m=e$; therefore $\text{R}=\dfrac{d}{e}$, and $\text{S}=\dfrac{c-d}{e}$.

II. *Solution by* S. WATSON; H. TOMLINSON; *and others.*



Let F, f be the foci, C the common centre of the axial and touching circles, the former cutting the curve in P, Q, the latter touching it in R. Put $\angle\,\text{QF}f=\alpha$, $\angle\,\text{PF}f=\beta$, $\text{F}f=2c=2ae$, and $\text{FC}=d$; then $r(a-c\cos\alpha)=b^2$,

therefore $r\cos\alpha=\dfrac{ar-b^2}{c}$. Similarly, $s\cos\beta=\dfrac{as-b^2}{c}$. But CQ = C

therefore $\qquad r^2+d^2-2rd\cos\alpha=s^2+d^2-2sd\cos\beta,$

therefore $\qquad r^2-s^2=2d\,(r\cos\alpha-s\cos\beta)=2d\,(r-s)\,\dfrac{a}{c}$

therefore $\qquad r+s=2\,\dfrac{ad}{c}=\dfrac{2d}{e}$ ....(1).

Now RC, being a normal at R, bisects the angle FRF; hence

$$\text{FR} : f\text{R} = \text{FC} : f\text{C},\text{ therefore FR} : d = 2a : 2ae,$$

therefore $\qquad \text{FR}=\dfrac{d}{e}=\tfrac{1}{2}(r+s)$, by (1).

Similarly, $f\text{R}=\frac{1}{2}(r'+s')$.

2237. (Proposed by M. W. CROFTON, B.A.)—Prove that the equation of the osculating circle at the extremity of the major axis of the ellipse $r+r'=2a$ is $\dfrac{r^2}{a+c}+\dfrac{r'^2}{a-c}=2a.$

I. *Solution by* W. CHADWICK; J. DALE; H. TOMLINSON; *and others.*

Since the equation of the ellipse is put in the form $r+r'=2a$, we have

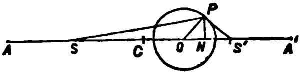
$$r=\left\{y^2+(x+c)^2\right\}^{\frac{1}{2}},\quad r'=\left\{y^2+(x-c)^2\right\}^{\frac{1}{2}}.$$

Now the radius of the osculating circle at the extremity of the major axis is $\dfrac{b^2}{a}$, and therefore the equation to this circle is

$$y^2+\left(x-\frac{c^2}{a}\right)^2=\frac{b^4}{a^2},\ \text{or}\ \frac{y^2+(x+c)^2}{a+c}+\frac{y^2+(x-c)^2}{a-c}=2a,$$

that is
$$\frac{r^2}{a+c}+\frac{r'^2}{a-c}=2a.$$

II. *Solution by* W. H. LAVERTY.



The centre of the osculating circle will evidently lie on the major axis. Then, first, to find the equation to a circle having its centre at the point $(r=k,\ r'=2c-k)$, we have, if $\text{SO}=k$, $\text{SC}=\text{CS}'=c$, $\text{OP}=\rho$, $\text{SP}=r$, $\text{S}'\text{P}=r'$,

$$\cos \text{SOP}+\cos \text{S}'\text{OP}=0,\quad\text{or}\quad \frac{k^2+\rho^2-r^2}{k}+\frac{(2c-k)^2+\rho^2-r'^2}{2c-k}=0;$$

whence we see that the equation to the circle may be written under the general form

$$kr^2+k'r'^2=1\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Now the osculating circle passes through the points

$$(r=a+c,\ r'=a-c),\quad (r=a+c-p,\ r'=a-c+p),$$

where p is a very small quantity; whence we have

$$k(a+c)^2+k'(a-c)^2=1;\ \text{and}\ k(a+c-p)^2+k'(a-c+p)^2=1.$$

Eliminating k, k', between these two equations and equation (1), we find

$$\frac{r^2}{a+c}+\frac{r'^2}{a-c}=2a.$$

ON FORMING THE MAGIC SQUARE OF NUMBERS from 1 to $(12n+6)^2$.

BY W. H. LAVERTY.

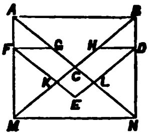
The magic square of an odd number $(6n+3)$ is by well-known rules easily attainable. Let P be the sum of any one of its rows, columns, or diagonals, so that

$$\text{P}=\frac{6n+3}{2}\left\{(6n+3)^2+1\right\}.$$

Now change (m) any one of the numbers from 1 to $(6n+3)^2$ into a small square of the form

$$\left.\begin{array}{l}(4m-1)\,.\,(4m-2)\\(4m-3)\,.\,(4m)\end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\alpha).$$

Now separate the constituents of the squares (α) by lines, so that the large square (A) becomes one having $\{(12n+6)\}$ numbers in each row; and let us examine what alterations we must make in order that (A) may be a magic square. Firstly, it is evident that the two *rows* of (α) have equal sums and consequently that the *rows* also of (A) will have equal sums. Secondly; in (α) the right hand column is greater by two than the left hand column; therefore in (A) the columns will be alternately greater and less than the proper sum $8P-3(6n+3)$ $(=Q$ suppose); by $\{(6n+3)\}$. Thirdly; one diagonal in (α) is less than the other by 4; therefore in (A) one will be less, and the other greater than Q by $2(6n+3)$. Now to remedy this; look at the figure; where $CH=CL=CK=CG=\frac{1}{3}CB=$ etc.; $BD=\frac{1}{3}BN=2n+1$. If in all the (α) squares, contained in the hexagon CAFEDBC; the places of $(4m)$ and $(4m-3)$ be interchanged; let us examine the result. Firstly; the *rows* do not change. Secondly; each of the too-small and too-great columns respectively in (A) has been increased or diminished by $\frac{6n+3}{3}\,.\,3$; the very result required. Also, the diagonals, which have been increased or diminished through $\frac{2}{3}$ of their length, have been increased or diminished by $\frac{2}{3}(6n+3)\,.\,3$; likewise the very result wanted. Hence we have a perfect magic square.



In the particular case of the following Question 2243, we have $n=0$; and the square stands as in the *first* of the arrangements there given. In the above figure (MB) is not a diagonal *line*, but a diagonal formed of (α) squares, similarly for the points B, D, &c.

2243. (Proposed by S. Bills.) — Place the numbers 1 to 36 in six rows, six in a row, so that each row, vertical, lateral, and diagonal, shall make 111.

Solution by W. H. Laverty; *and the* Proposer.

Of the two following arrangements, the *first* may be obtained by making $n=0$ in the foregoing paper;—

31	30	3	2	23	22
32	29	1	4	24	21
11	10	19	18	27	26
9	12	20	17	25	28
15	14	35	34	7	6
18	16	33	36	5	8

5	32	3	34	1	36
12	29	10	27	8	25
15	20	23	13	19	21
18	17	14	24	22	16
28	7	30	11	26	9
3	6	31	2	35	4

NOTE ON QUESTION 1990. BY M. W. CROFTON, B.A.

The theorem in paragraph 4 will be found stated as a note in a paper on the Cartesian ovals which I have given in the *Proceedings of the London Mathematical Society*, No. VI. The theorems ennunciated by Professor Cayley in connexion with it (*Reprint*, Vol. VIII., p. 51) will be found (in substance), with their proofs, in the text of the above Paper. That αP, αP_1 are tangents to one Cartesian, and $\alpha' P$, $\alpha' P_1$ to the other, is an evident consequence of the theorem in Quest. 1924 (*Educational Times* for March, 1866):—that a Cartesian is always equally inclined to the circle through any two foci and the straight line through the third.

These theorems are cases of the following more general ones regarding bicircular quartics:—

If two given circles meet in P, P_1, and they be cut by any arbitrary circle in four points 1234; then the two (complete) bicircular quartics, foci 1234, which pass through P, also pass through P_1; and if α, α' be the two centres of similitude of the given circles, then αP, αP_1 will be tangents to one bicircular quartic; $\alpha' P$, $\alpha' P_1$ those to the other; these tangents being thus quite independent of the positions of the foci 1234 on the two given circles.

Hence, given two circles, the four points where any arbitrary circle meets them are the foci of one of a system of bicircular quartics which have double contact with each other at the two intersections of the given circles.

Several results connected with these theorems are given in a paper which I presented to the London Mathematical Society at one of their recent meetings.

ON A CRITERION OF DIVISIBILITY BY 7 AND 13.

BY M. JENKINS, B.A.

In the *Proceedings of the Royal Society* (Vol. VII., No. 3, p. 42, Ap. 27, 1854,) it is noticed by the Rev. James Booth, LL.D., F.R.S., that a number of six places of figures, consisting of a repetition of a period of any three figures, is divisible by 7 and 13. Example 507507.

A useful criterion of divisibility by these two numbers is thus suggested. It is given in the following rule by making $p=7$ or 13, $r=10$, and therefore $t=6$, $\frac{1}{2}t=3$.

Let r be the scale of numbers, p any uneven prime number, t the index to which r appertains, and suppose that t is even.

Then to find out if any number of more than $\frac{1}{2}t$ digits is divisible by p.

"Add sufficient cyphers to the left hand of the number to make its number of digits a multiple of $\frac{1}{2}t$; subtract the extreme left-hand digit from the next $\frac{1}{2}t$th digit, and the residue from the next $\frac{1}{2}t$th digit, and so on until we obtain a final residue; form similar final residues beginning with the second, third, &c. digits from the left hand; then the divisibility by p of the number formed by the $\frac{1}{2}t$ final residues taken in order is a criterion of the divisibility by p of the given number."

The subtraction must be algebraical, and therefore the criterion number may contain negative as well as positive digits. The process gives a number which is a criterion of divisibility by every prime divisor of $r^{\frac{1}{2}t}+1$; if, however, we are considering only a single prime divisor p, the final residues may be made positive by the addition of any multiple of p during the process, and may also be made less than r if p be less than r.

Examples : 57,398,362 is divisible by 7 because 021 is divisible by 7 and leaves a remainder 8 (mod. 13).

The criterion number of 645,268 is (−4, 2, 3) or 377 for both 7 and 13; or 923 for 13 or 323 for 7.

A similar criterion exists when t is odd; t taking the place of $\frac{1}{2}t$ and addition being used instead of subtraction.

The proof of the correctness of the criteria is simple, depending on the facts that $r^{\frac{1}{2}t}+1$ in the first case, and $r^{t}-1$ in the latter is divisible by p.

A *sufficient* condition for t being even when $r=10$ and p is prime, is $p \equiv \pm 7, \ \pm 11, \ \pm 17$, or ± 19 (mod. 40), since in that case, by a theorem of Euler's, $10^{\frac{1}{2}(p-1)} \equiv -1$ (mod. p), and therefore t is a divisor of $p-1$ contaning the same power of 2.

The antitheses with respect to the nature of the recurring r-mal of p^{-1} are shown below.

t even.

(I.) A number is divisible by p if it contains twice ($\frac{1}{2}t$) digits *repetitive*, i.e. $\frac{1}{2}t$ and $\frac{1}{2}t$ respectively in direct order.	(II.) Period of p^{-1} in scale r consists of twice ($\frac{1}{2}t$) digits *complementary* to $r-1$.

t odd.

(III.) twice (t) digits *complementary* to p.	(IV.) t figures non-complementary, or twice (t) figures *repetitive.*

NOTE ON THE FOUR-POINT PROBLEM. BY W. S. B. WOOLHOUSE, F.R.A.S.

An approximate solution of this problem may be obtained as follows:—

It is just the same whether four points A, B, C, D are taken at random, or two lines AB, CD are taken at random. In the latter case, the relative position of the lines as regards intersection evidently resolves into three cases:—

1. The lines may already intersect;
2. The lines may intersect on the production of one of them;
3. The lines may intersect on the production of both.

The apparent probabilities of these several cases, on consideration, appear to be nearly equal, and the second alone is that in which the quadrilateral is reentrant. This probability is therefore $\frac{1}{3}$, at least approximately. It is here unnecessary to assume any boundary.

1911. (Proposed by Professor CAYLEY.)—Given four points, and also the "conic of centres"—viz. the conic which is the locus of the centres of the several conics which pass through the four given points; then if a conic through the four given points has for its centre a given point on the conic of centres, it is required to find a construction for the asymptotes of this conic.

I. *Solution by the* PROPOSER.

1. Consider four given points, and in connection therewith a given line IJ; the locus of the poles of IJ, in regard to the several conics which pass through the four points, is a conic, the "conic of poles." Consider a particular conic Θ, through the four points; the pole of IJ in regard to the conic Θ is a point C on the conic of poles, and the tangents from C to the conic Θ meet the conic of poles in two points H, K; the chord of intersection HK passes through the point Π which is the pole of IJ in regard to the conic of poles. Moreover, the polars of a point C′, in regard to the several conics through the four points, meet in a point Ω′, the "common pole" of C′, and in particular if C′ be the point C on the conic of poles, then the common pole is a point Ω on the line IJ; this being so, the line HK passes (as already mentioned) through Π, and the lines HK and ΠΩ are harmonics in regard to the conic of poles.

2. Assuming the foregoing properties, then, given the four points, the line IJ, the conic of poles, and the point C on this conic; we may construct Π the pole of IJ in regard to the conic of poles; and also Ω the common pole of C; the line HK is then given as a line passing through Π, and harmonic to ΠΩ in regard to the conic of poles; this line meets the conic of poles in the points H, K; and then CH, CK are the tangents from C to a conic Θ which passes through the four points.

3. In particular if IJ be the line infinity, then the conic of poles is the conic of centres; Π is the centre of this conic; Ω is as before the common pole of C; HK is given as the diameter of the conic of centres, conjugate to ΠΩ; H, K are the extremities of this diameter; and then CH, CK are the asymptotes of the conic through the four points, which has the point C for its centre; and the asymptotes are therefore constructed as required. If the points H, K are imaginary, the asymptotes will be also imaginary; the conic Θ is in this case an ellipse.

4. It is hardly necessary to remark, in regard to the construction of the point Ω, that we have among the conics through the four points, three pairs of lines meeting in points P, Q, R respectively (it is clear that the conic of poles passes through these three points); the harmonics of CP, CQ, CR in regard to the three pairs of lines respectively meet in a point, which is the required point Ω. In the particular case where the point C is on the conic of centres, the three harmonics are parallel; it is therefore sufficient to construct *one* of them; and the line HK is then the diameter of the conic of poles, conjugate to the harmonic so constructed.

5. It remains to prove the properties assumed in (1). We may take $z = 0$ for the equation of the line IJ, $x = 0$, $y = 0$ for the equations of the tangents to the conic Θ at its intersections with the line IJ, so that we have $(x=0, y=0)$ for the coordinates of the point C; the equation of the conic Θ will be of the form $z^2 - xy = 0$, and the four points may then be taken to be the intersections of the conic $z^2 - xy = 0$, and the arbitrary conic

$$(a, b, c, f, g, h)(x, y, z)^2 = 0.$$

The equation of the conic of centres is found to be

$$x\,(ax + hy + gz) - y\,(hx + by + fz) = 0, \text{ or } ax^2 - by^2 + gzx - hxy = 0;$$

or, as it may also be written,

$$(2a, -2b, 0, -f, g, 0)\ (x, y, z)^2 = 0;$$

and it is convenient to remark that the equation in line coordinates (or condition that this conic may be touched by the line $\xi x + \eta y + \zeta z = 0$) is

$$(-f^2, -g^2, -4ab, 2af, 2bg, -fg)\ (\xi, \eta, \zeta)^2 = 0.$$

The line $x = 0$ meets the conic of poles in the point $x = 0$, $by + fz = 0$, and the line $y = 0$ meets the same conic in the point $y = 0$, $ax + gz = 0$; hence the line HK, which is the line joining these two points, has for its equation

$$afx + bgy + fgz = 0;$$

and it only remains to be shown that this line passes through the point Π, and is the harmonic of the line $\Pi\Omega$ in regard to the conic of centres. The point Π is the pole of the line $z = 0$ in regard to the conic of centres, its coordinates are at once found to be

$$x : y : z = bg : af : -2ab;$$

and we thence see that Π is a point on the line HK. The point Ω is given as the intersection of the polars of C in regard to the conics $z^2 - xy = 0$, and $(a, b, c, f, g, h)(x, y, z)^2 = 0$ respectively; that is, as the intersection of the lines $z = 0$, and $gx + fy + cz = 0$; its coordinates therefore are

$$x : y : z = -f : g : 0.$$

Hence the equation of the line $\Pi\Omega$ is

$$2abg\,x + 2abf\,y + (af^2 + bg^2)\,z = 0.$$

Now, in general, if we have a conic the line-equation whereof is $(\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{F}, \mathrm{G}, \mathrm{H})(\xi, \eta, \zeta)^2 = 0$, then the condition in order that, in regard thereto, the lines $\lambda x + \mu y + \nu z = 0$ and $\lambda' x + \mu' y + \nu' z = 0$ may be harmonics, is

$$(\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{F}, \mathrm{G}, \mathrm{H})(\lambda, \mu, \nu)(\lambda', \mu', \nu') = 0;$$

that is,

$$\mathrm{A}\lambda\lambda' + \mathrm{B}\mu\mu' + \mathrm{C}\nu\nu' + \mathrm{F}(\mu\nu' + \mu'\nu) + \mathrm{G}(\nu\lambda' + \nu'\lambda) + \mathrm{H}(\lambda\mu' + \lambda'\mu) = 0.$$

Hence, in order that the two lines HK and $\Pi\Omega$ may be harmonics in regard to the conic of centres, we should have

$$(-f^2, -g^2, -4ab, 2af, 2bg, -fg)(af, bg, fg)(2abg, 2abf, af^2 + bg^2) = 0.$$

But developing, and omitting the common factor $abfg$, which enters into all the terms, this equation is

$$-(2af^2) - (2bg^2) - 4(af^2 + bg^2) + \left\{4af^2 + 2(af^2 + bg^2)\right\}$$
$$+ \left\{4bg^2 + 2(af^2 + bg^2)\right\} - 2(af^2 + bg^2) = 0,$$

which is identically true; and the lines HK and $\Pi\Omega$ are therefore harmonics in regard to the conic of centres.

II. *Solution by the* Rev. R. Townsend, F.R.S.

Professor Cayley's question is evidently equivalent to the following:—"Given the centre O of a conic, and three points A, B, C on the curve, to determine the directions, real or imaginary, of the asymptotes"—which may be solved immediately as follows:—If X, Y, Z be the three points at infinity on, and U, V, W the three middle points of, the three sides BC, CA, AB of the triangle ABC; then, the three pairs of lines OX and OU, OY and OV, OZ and OW being evidently pairs of conjugate diameters of the conic, the two double rays, real or imaginary, OM and ON, of the involution they determine, as dividing harmonically the three angles they form, are consequently the two asymptotes required.

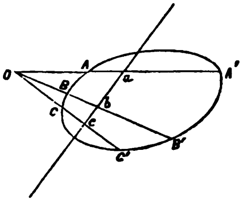
The more general problem—"Given a point and line P and L pole and polar with respect to a conic, and three points A, B, C on the curve, to determine the directions, real or imaginary, of the two tangents from P to the curve"—may be solved in precisely the same manner. For, if X, Y, Z be the three intersections of L with, and U, V, W the three points harmonically conjugate to, them with respect to the three sides BC, CA, AB of the triangle ABC; then, the three pairs of lines PX and PU, PY and PV, PZ and PW being evidently pairs of conjugate lines with respect to the conic, the two double rays, real or imaginary, PM and PN, of the involution they determine, as dividing harmonically the three angles they form, are consequently the two tangents required.

The reciprocal problem—"Given a point and line P and L pole and polar with respect to a conic, and three tangents A, B, C to the curve, to determine the positions, real or imaginary, of the two intersections of L with the curve," with the particular case of it—"Given the centre O of a conic and three tangents A, B, C to the curve, to determine the directions, real or imaginary, of the two points at infinity,"—may of course be solved by the reciprocal of the preceding construction.

III. *Solution by* F. D. Thomson, M.A.

In the proposed problem, we know the centre of the conic and four points through which the conic passes, and we have to draw the tangents to the conic from the centre. The problem is, therefore, a particular case of the following:—"Given a point and its polar with respect to a conic passing through three given points, to draw the tangent through the given point." The more general problem may be solved as follows:—

Let O be the given point, *abc* its polar, A, B, C the given points; then since the polar *abc* divides harmonically the chord through O, the points A′, B′, C′, where OA, OB, OC meet the curve again, are known.



Also (C . ABA′B′) = (C′ . ABA′B′).

Take the segment on *abc*; then we get two homographic sections $\alpha\beta\alpha'\beta'$, $\alpha_1\beta_1\alpha_1'\beta_1'$, suppose, the *double points* of which will be the points where *abc* meets the curve, that is, the points of contact of tangents through O.

In the particular case proposed, *abc* is the line at infinity, and $OA' = -OA$, $OB' = -OB$, $OC' = -OC$; and if we draw lines $O\alpha$, $O\beta$, $O\alpha'$, $O\beta'$ . . . . parallel to CA, CB, CA′, CB′ respectively, and $O\alpha_1$, $O\beta_1$, $O\alpha_1'$, $O\beta_1'$ parallel to C′A, C′B, C′A′, C′B′. . . . . . ; then $(O \,.\, \alpha\beta\alpha'\beta' \ldots .)$, $(O \,.\, \alpha_1\beta_1\alpha'_1\beta'_1 \ldots .)$ form two homographic pencils, the double rays of which will be the asymptotes required.

2378. (Proposed by the Editor.)—Find the average (1) of the areas of all the circles which can be drawn within the circumference of a given circle, and (2) of the volumes of all the spheres which can be drawn within the surface of a given sphere; also find the probability (3) that any one of the circles taken at random from (1) will exceed a given circle, and (4) that any one of the spheres taken at random from (2) will exceed a given sphere.

Solution by S. BILLS ; S. WATSON ; *and others.*

1. Let a be the radius of the given circle, r the radius of a variable circle, x the distance of its centre from the centre of the given circle, N the number of circles that can be drawn within the given circle, S the sum of all their areas, and A the average required. Then we have

$$N = \int_0^a (a-x)\,.\,2\pi x dx = \tfrac{1}{3}\pi a^3,\ S = \int_0^a 2\pi x dx \int_0^{a-x} \pi r^2 dr = \tfrac{1}{30}\pi^2 a^4;$$

therefore $$A = \frac{S}{N} = \frac{\pi a^2}{10} = \tfrac{1}{10}\text{ (area of given circle).}$$

2. Adopting a similar notation for the sphere, we have

$$N = \int_0^a (a-x)\,.\,4\pi x^2 dx = \tfrac{1}{3}\pi a^4,\ S = \int_0^a 4\pi x^2 dx \int_0^{a-x} \tfrac{4}{3}\pi r^3 dr = \tfrac{4}{315}\pi^2 a^7;$$

therefore $$A = \frac{S}{N} = \tfrac{4}{105}\pi a^3 = \tfrac{1}{35}\text{ (volume of given sphere).}$$

3. Let N_1 be the number of circles, greater than a circle of radius c, that can be drawn within the given circle ; then we have

$$N_1 = \int_0^{a-c} (a-x-c)\,.\,2\pi x dx = \tfrac{1}{3}\pi (a-c)^3;$$

hence the probability (p) that a circle drawn at random in (1) will have a radius greater than c is

$$p = \frac{N_1}{N} = \left(\frac{a-c}{a}\right)^3.$$

4. Using the same notation for the sphere, we have

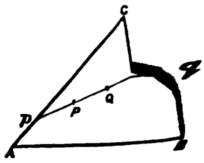
$$N_1 = \int_0^{a-c} (a-x-c)\,.\,4\pi x^2 dx = \tfrac{1}{3}\pi\,(a-c)^4;$$

therefore, in this case, the required probability is

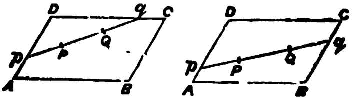
$$p = \frac{N_1}{N} = \left(\frac{a-c}{a}\right)^4.$$

2371. (Proposed by Professor CAYLEY.)—If P, Q be two points tak at random within the triangle ABC, what is the chance that the poi A, B, P, Q may form a convex quadrangle ?

Solution by the REV. M. M. U. WILKINSON.



1. Produce PQ both ways till it meets the periphery of the triangle in p, q. If AB be the side on which lies neither p nor q, APQB will be convex, but APQC, BPQC will be reentrant. Therefore the chance of the quadrilateral being reentrant is twice the chance of its being convex, and is therefore $\tfrac{2}{3}$.



2. Taking the case of a parallelogram on AB, the same method shows that, if the sides, in which p and q lie, meet in D, then DAPQ, DCPQ are reentrant, and BAPQ, CBPQ convex. (APQC, BPQD are not included in the conditions of the problem.) If the sides, in which p and q lie, are parallel, as AD, BC, then DPQC, APQB are convex, and DAPQ, BPQC reentrant. The chance of reentrance is therefore $\frac{1}{2}$.

2416. (Proposed by the Rev. R. Townsend, F.R.S.)—Find the point in a plane the sum of the squares of whose distances from any number of given lines in the plane shall be the minimum.

Solution by James Dale.

Let (l_1, m_1, n_1), (l_2, m_2, n_2), ... (l_n, m_n, n_n) be the given lines, $d_1, d_2, d_3 \ldots\ldots d_n$ the distances of the required point (x, y, z) from these n lines; then

$$d_1^2 = \frac{(l_1x + m_1y + n_1z)^2}{\{l_1, m_1, n_1\}^2},\ d_2^2 = \frac{(l_2x + m_2y + n_3z)^2}{\{l_2, m_2, n_3\}^2}, \ldots\ldots$$

$$d_n^2 = \frac{(l_nx + m_ny + n_nz)^2}{\{l_n, m_n, n_n\}^2}, \text{ where}$$

$$\{l_r, m_r, n_r\}^2 = l_r^2 + m_r^2 + n_r^2 - 2m_rn_r \cos \mathrm{A} - 2n_rl_r \cos \mathrm{B} - 2l_rm_r \cos \mathrm{C}.$$

Adding these, and equating the differential of the sum to zero, we get

$$\left\{ \Sigma\left(\frac{l^2}{\{l, m, n\}^2}\right) x + \Sigma\left(\frac{lm}{\{l, m, n\}^2}\right) y + \Sigma\left(\frac{ln}{\{l, m, n\}^2}\right) z \right\} dx$$

$$+ \ldots\ldots dy + \ldots\ldots dz = 0,$$

and as $ax + by + cz = 2\Delta$, we have $adx + bdy + cdz = 0$;

$$\therefore \frac{\Sigma\left(\frac{l^2}{\{l, m, n\}^2}\right) x + \Sigma\left(\frac{lm}{\{l, m, n\}^2}\right) y + \Sigma\left(\frac{ln}{\{l, m, n\}^2}\right) z}{a}$$

$$= \frac{\Sigma\left(\frac{ml}{\{l, m, n\}^2}\right) x + \Sigma\left(\frac{m^2}{\{l, m, n\}^2}\right) y + \Sigma\left(\frac{mn}{\{l, m, n\}^2}\right) z}{b}$$

$$= \frac{\Sigma\left(\frac{nl}{\{l, m, n\}^2}\right) x + \Sigma\left(\frac{nm}{\{l, m, n\}^2}\right) y + \Sigma\left(\frac{n^2}{\{l, m, n\}^2}\right) z}{c}.$$

These equations, together with $ax + by + cz = 2\Delta$, determine x, y, z.
When the number of lines is three, taking them for the sides of the triangle of reference, the equations reduce to $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$.

2261. (Proposed by S. Bills.)—Sixty-three school-boys walk out three abreast for thirty-one days in succession; it is required to arrange them daily so that no two of them shall walk twice abreast.

Solution by W. S. B. Woolhouse, F.R.A.S.

Let the sixty-three school-boys be denoted by k; 1, (1); 2, (2); 3, (3); ... 31, (31); and let us endeavour so to arrange these symbols in threes that, after performing a (31) cyclical operation, by successively increasing the numerals by unity, no pair of symbols shall be repeated, and the conditions of the question shall thus be realized. As an auxiliary let r be a primitive root to modulus 31; that is, a number such that the successive powers $r^0, r, r^2, r^3, \dots r^{29}$ when severally divided by 31 shall all leave different remainders. Then these thirty remainders must necessarily exhibit all the consecutive numbers 1 to 30. In what follows let it be distinctly understood that all values employed are to be estimated according to this modulus, and that the number 31 and all multiples of it are to be rejected from all values that ordinarily exceed such amount. According to the even and odd powers of r, it will be convenient to partition the numbers 1 to 30 in two systems, viz.,

$$\left.\begin{array}{ll} \text{Even system} & r^0, r^2, r^4, \dots\dots r^{28} \\ \text{Odd system} & r, r^3, r^5, \dots\dots r^{29} \end{array}\right\} \dots(\theta).$$

Suppose now that the sixty-three symbols have for their primitive arrangement either of the two following forms A and B:—

A			B		
k	31	(31)	k	31	(31)
r	r^{11}	r^{21}	r^0	r^{10}	r^{20}
r^3	r^{13}	r^{23}	r^2	r^{12}	r^{22}
r^5	r^{15}	r^{25}	r^4	r^{14}	r^{24}
r^7	r^{17}	r^{27}	r^6	r^{16}	r^{26}
r^9	r^{19}	r^{29}	r^8	r^{18}	r^{28}
r^0	(r^{α})	(r^{β})	r	$(r^{\alpha+1})$	$(r^{\beta+1})$
r^2	$(r^{\alpha+2})$	$(r^{\beta+2})$	r^3	$(r^{\alpha+3})$	$(r^{\beta+3})$
r^4	$(r^{\alpha+4})$	$(r^{\beta+4})$	r^5	$(r^{\alpha+5})$	$(r^{\beta+5})$
⋮	⋮	⋮	⋮	⋮	⋮
r^{28}	$(r^{\alpha+28})$	$(r^{\beta+28})$	r^{29}	$(r^{\alpha+29})$	$(r^{\beta+29})$.

The only possibility of repetition of symbols or duads would be amongst the last 15 triads of symbols which are placed below the second horizontal rule. Of these each vertical column will evidently constitute one of the two systems (θ) in a direct order of succession. Those included within parentheses, that is the second and third vertical columns, must be made to form different systems, and therefore α and β must be unlike as to even and odd.

Again, if the numbers in the first column be subtracted from those in the second and third, we get two sets of differences, viz.,

$$\begin{array}{cc|cc}
\multicolumn{2}{c|}{A} & \multicolumn{2}{c}{B} \\
r^{\alpha}-r^{0} & r^{\beta}-r^{0} & r^{\alpha+1}-r & r^{\beta+1}-r \\
r^{\alpha+2}-r^{2} & r^{\beta+2}-r^{2} & r^{\alpha+3}-r^{3} & r^{\beta+3}-r^{3} \\
r^{\alpha+4}-r^{4} & r^{\beta+4}-r^{4} & r^{\alpha+5}-r^{5} & r^{\beta+5}-r^{5} \\
\vdots \quad \vdots & \vdots \quad \vdots & \vdots \quad \vdots & \vdots \quad \vdots \\
r^{\alpha+28}-r^{28} & r^{\beta+28}-r^{28} & r^{\alpha+29}-r^{29} & r^{\beta+29}-r^{29}
\end{array}$$

and it is evident that in the cyclical process no repetition of a duad can occur unless values in one of these two sets of differences should be identical with values contained in the other set. But each of these sets of differences progressing by the common ratio r^2 must necessarily also constitute one of the two systems (θ), and hence the values will be all different if each set appertains to a different system. Hence if we make $r^{\alpha}-1 = r^{\alpha'}$ and $r^{\beta}-1 = r^{\beta'}$ the indices α', β' must be unlike as to even and odd.

We have therefore only to comply with the two following requirements:

α, β to be unlike as to even and odd, and neither of them to be zero;

α', β' also to be unlike as to even and odd.

The number 31 has four reciprocal pairs of primitive roots, viz., (3, 21), (11, 17), (12, 13), (22, 24). To work out the calculation numerically, take $r = 3$, and we readily form the following table showing the values of r^{α}, r^{β}, α', β' classed in two divisions according to the even and odd systems.

α	r^{α}	α'	α	r^{α}	α'
0	1	..	1	3	24
2	9	12	3	27	5
4	19	26	5	26	10
6	16	21	7	17	6
8	20	4	9	29	16
10	25	13	11	13	19
12	8	28	13	24	27
14	10	2	15	30	9
16	28	3	17	22	29
18	4	1	19	12	23
20	5	18	21	15	22
22	14	11	23	11	14
24	2	30	25	6	20
26	18	7	27	23	17
28	7	25	29	21	8
β	r^{β}	β'	β	r^{β}	β'

By inspecting this table the suitable cooperating values of α, β, which satisfy the foregoing requirements, are readily extracted as follows:—

$$\begin{cases} \alpha \text{ even}, \ \alpha' \text{ even}, & \alpha = 2, \ 4, \ 8, 12, 14, 20, 24 \\ \beta \text{ odd}, \ \beta' \text{ odd}, & \beta = 3, 11, 13, 15, 17, 19, 27 \end{cases}$$

$$\begin{cases} \alpha \text{ odd}, \ \alpha' \text{ even}, & \alpha = 1, \ 5, \ 7, \ 9, 21, 23, 25, 29 \\ \beta \text{ even}, \ \beta' \text{ odd}, & \beta = 6, 10, 16, 18, 22, 26, 28 \end{cases}.$$

The corresponding values of r^α, r^β are

$$\begin{cases} r^\alpha = \ 9, 19, 20, \ 8, 10, \ 5, \ 2 \\ r^\beta = 27, 13, 24, 30, 22, 12, 23 \end{cases}$$

$$\begin{cases} r^\alpha = \ 3, 26, 17, 29, 15, 11, \ 6, 21 \\ r^\beta = 16, 25, 28, \ 4, 14, 18, \ 7 \end{cases},$$

and in either of these cases (r^α) (r^β) may be any number contained in the first line associated with any number in the second line. The number of sets of these values is therefore $7.7 + 8.7 = 105$; and as the forms A and B may be both resorted to with each of them, the number of distinct constructions is 210.

As an example, take $\alpha = 2$, $\beta = 11$, or $r^\alpha = 9$, $r^\beta = 13$; then the primitive arrangements A and B, in which the several numbers are filled in vertically from the preceding table, are the ollowing:—

A			B		
h	31	(31)	h	31	(31)
3	13	15	1	25	5
27	24	11	9	8	14
26	30	6	19	10	2
17	22	23	16	28	18
29	12	21	20	4	7
1	(9)	(13)	3	(27)	(8)
9	(19)	(24)	27	(26)	(10)
19	(16)	(30)	26	(17)	(28)
16	(20)	(22)	17	(29)	(4)
20	(25)	(12)	29	(13)	(5)
25	(8)	(15)	13	(24)	(14)
8	(10)	(11)	24	(30)	(2)
10	(28)	(6)	30	(22)	(18)
28	(4)	(23)	22	(12)	(7)
4	(5)	(21)	12	(15)	(1)
5	(14)	(3)	15	(11)	(9)
14	(2)	(27)	11	(6)	(19)
2	(18)	(26)	6	(23)	(16)
18	(7)	(17)	23	(21)	(20)
7	(1)	(29)	21	(3)	(25)

The cyclical development of the first of these, showing all the required thirty-one arrangements, is produced by successively increasing all the numbers by an unit, and rejecting 31 when necessary, as the following will show:—

h	31	(31)	*h*	1	(1)	*h*	2	(2)	*h*	3	(3)	*h*	4	(4)
3	13	15	4	14	16	5	15	17	6	16	18	7	17	19
27	24	11	28	25	12	29	26	13	30	27	14	31	28	15
26	30	6	27	31	7	28	1	8	29	2	9	30	3	10
17	22	23	18	23	24	19	24	25	20	25	26	21	26	27
29	12	21	30	13	22	31	14	23	1	15	24	2	16	25
1	(9)	(13)	2	(10)	(14)	3	(11)	(15)	4	(12)	(16)	5	(13)	(17)
9	(19)	(24)	10	(20)	(25)	11	(21)	(26)	12	(22)	(27)	13	(23)	(28)
19	(16)	(30)	20	(17)	(31)	21	(18)	(1)	22	(19)	(2)	23	(20)	(3)
16	(20)	(22)	17	(21)	(23)	18	(22)	(24)	19	(23)	(25)	20	(24)	(26)
20	(25)	(12)	21	(26)	(13)	22	(27)	(14)	23	(28)	(15)	24	(29)	(16)
25	(8)	(15)	26	(9)	(16)	27	(10)	(17)	28	(11)	(18)	29	(12)	(19)
8	(10)	(11)	9	(11)	(12)	10	(12)	(13)	11	(13)	(14)	12	(14)	(15)
10	(28)	(6)	11	(29)	(7)	12	(30)	(8)	13	(31)	(9)	14	(1)	(10)
28	(4)	(23)	29	(5)	(24)	30	(6)	(25)	31	(7)	(26)	1	(8)	(27)
4	(5)	(21)	5	(6)	(22)	6	(7)	(23)	7	(8)	(24)	8	(9)	(25)
5	(14)	(3)	6	(15)	(4)	7	(16)	(5)	8	(17)	(6)	9	(18)	(7)
14	(2)	(27)	15	(3)	(28)	16	(4)	(29)	17	(5)	(30)	18	(6)	(31)
2	(18)	(26)	3	(19)	(27)	4	(20)	(28)	5	(21)	(29)	6	(22)	(30)
18	(7)	(17)	19	(8)	(18)	20	(9)	(19)	21	(10)	(20)	22	(11)	(21)
7	(1)	(29)	8	(2)	(30)	9	(3)	(31)	10	(4)	(1)	11	(5)	(2)

h	5	(5)	*h*	6	(6)	*h*	7	(7)	*h*	8	(8)	*h*	9	(9)
8	18	20	9	19	21	10	20	22	11	21	23	12	22	24
1	29	16	2	30	17	3	31	18	4	1	19	5	2	20
31	4	11	1	5	12	2	6	13	3	7	14	4	8	15
22	27	28	23	28	29	24	29	30	25	30	31	26	31	1
3	17	26	4	18	27	5	19	28	6	20	29	7	21	30
6	(14)	(18)	7	(15)	(19)	8	(16)	(20)	9	(17)	(21)	10	(18)	(22)
14	(24)	(29)	15	(25)	(30)	16	(26)	(31)	17	(27)	(1)	18	(28)	(2)
24	(21)	(4)	25	(22)	(5)	26	(23)	(6)	27	(24)	(7)	28	(25)	(8)
21	(25)	(27)	22	(26)	(28)	23	(27)	(29)	24	(28)	(30)	25	(29)	(31)
25	(30)	(17)	26	(31)	(18)	27	(1)	(19)	28	(2)	(20)	29	(3)	(21)
30	(13)	(20)	31	(14)	(21)	1	(15)	(22)	2	(16)	(23)	3	(17)	(21)
13	(15)	(16)	14	(16)	(17)	15	(17)	(18)	16	(18)	(19)	17	(19)	(20)
15	(2)	(11)	16	(3)	(12)	17	(4)	(13)	18	(5)	(11)	19	(6)	(15)
2	(9)	(28)	3	(10)	(29)	4	(11)	(30)	5	(12)	(31)	6	(13)	(1)
9	(10)	(26)	10	(11)	(27)	11	(12)	(28)	12	(13)	(29)	13	(14)	(30)
10	(19)	(8)	11	(20)	(9)	12	(21)	(10)	13	(22)	(11)	14	(23)	(12)
19	(7)	(1)	20	(8)	(2)	21	(9)	(3)	22	(10)	(4)	23	(11)	(5)
7	(23)	(31)	8	(24)	(1)	9	(25)	(2)	10	(26)	(3)	11	(27)	(4)
23	(12)	(22)	24	(13)	(23)	25	(14)	(24)	22	(15)	(25)	27	(16)	(26)
12	(6)	(3)	13	(7)	(4)	14	(8)	(5)	15	(9)	(6)	16	(10)	(7)

h	10	(10)	*h*	11	(11)	*h*	12	(12)	*h*	13	(13)	*h*	14	(14)
13	23	25	14	24	26	15	25	27	16	26	28	17	27	29
6	3	21	7	4	22	8	5	23	9	6	24	10	7	25
5	9	16	6	10	17	7	11	18	8	12	19	9	13	20
27	1	2	28	2	3	29	3	4	30	4	5	31	5	6
8	22	31	9	23	1	10	24	2	11	25	3	12	26	4
11	(19)	(23)	12	(20)	(24)	13	(21)	(25)	14	(22)	(26)	15	(23)	(27)
19	(29)	(3)	20	(30)	(4)	21	(31)	(5)	22	(1)	(6)	23	(2)	(7)
29	(26)	(9)	30	(27)	(10)	31	(28)	(11)	1	(29)	(12)	2	(30)	(13)
26	(30)	(1)	27	(31)	(2)	28	(1)	(3)	29	(2)	(4)	30	(3)	(5)
30	(4)	(22)	31	(5)	(23)	1	(6)	(24)	2	(7)	(25)	3	(8)	(26)
4	(18)	(25)	5	(19)	(26)	6	(20)	(27)	7	(21)	(28)	8	(22)	(29)
18	(20)	(21)	19	(21)	(22)	20	(22)	(23)	21	(23)	(24)	22	(24)	(25)
20	(7)	(16)	21	(8)	(17)	22	(9)	(18)	23	(10)	(19)	24	(11)	(20)
7	(14)	(2)	8	(15)	(3)	9	(16)	(4)	10	(17)	(5)	11	(18)	(6)
14	(15)	(31)	15	(16)	(1)	16	(17)	(2)	17	(18)	(3)	18	(19)	(4)
15	(24)	(13)	16	(25)	(14)	17	(26)	(15)	18	(27)	(16)	19	(28)	(17)
24	(12)	(6)	25	(13)	(7)	26	(14)	(8)	27	(15)	(9)	28	(16)	(10)
12	(28)	(5)	13	(29)	(6)	14	(30)	(7)	15	(31)	(8)	16	(1)	(9)
28	(17)	(27)	29	(18)	(28)	30	(19)	(29)	31	(20)	(30)	1	(21)	(31)
17	(11)	(8)	18	(12)	(9)	19	(13)	(10)	20	(14)	(11)	21	(15)	(12)

h	15	(15)	*h*	16	(16)	*h*	17	(17)	*h*	18	(18)	*h*	19	(19)
18	28	30	19	29	31	20	30	1	21	31	2	22	1	3
11	8	26	12	9	27	13	10	28	14	11	29	15	12	30
10	14	21	11	15	22	12	16	23	13	17	24	14	18	25
1	6	7	2	7	8	3	8	9	4	9	10	5	10	11
13	27	5	14	28	6	15	29	7	16	30	8	17	31	9
16	(24)	(28)	17	(25)	(29)	18	(26)	(30)	19	(27)	(31)	20	(28)	(1)
24	(3)	(8)	25	(4)	(9)	26	(5)	(10)	27	(6)	(11)	28	(7)	(12)
3	(31)	(14)	4	(1)	(15)	5	(2)	(16)	6	(3)	(17)	7	(4)	(18)
31	(4)	(6)	1	(5)	(7)	2	(6)	(8)	3	(7)	(9)	4	(8)	(10)
4	(9)	(27)	5	(10)	(28)	6	(11)	(29)	7	(12)	(30)	8	(13)	(31)
9	(23)	(30)	10	(24)	(31)	11	(25)	(1)	12	(26)	(2)	13	(27)	(3)
23	(25)	(26)	24	(26)	(27)	25	(27)	(28)	26	(28)	(29)	27	(29)	(30)
25	(12)	(21)	26	(13)	(22)	27	(14)	(23)	28	(15)	(24)	29	(16)	(25)
12	(19)	(7)	13	(20)	(8)	14	(21)	(9)	15	(22)	(10)	16	(23)	(11)
19	(20)	(5)	20	(21)	(6)	21	(22)	(7)	22	(23)	(8)	23	(24)	(9)
20	(29)	(18)	21	(30)	(19)	22	(31)	(20)	23	(1)	(21)	24	(2)	(22)
29	(17)	(11)	30	(18)	(12)	31	(19)	(13)	1	(20)	(14)	2	(21)	(15)
17	(2)	(10)	18	(3)	(11)	19	(4)	(12)	20	(5)	(13)	21	(6)	(14)
2	(22)	(1)	3	(23)	(2)	4	(24)	(3)	5	(25)	(4)	6	(26)	(5)
22	(16)	(13)	23	(17)	(14)	24	(18)	(15)	25	(19)	(16)	26	(20)	(17)

h 20 (20)			*h* 21 (21)			*h* 22 (22)			*h* 23 (23)			*h* 24 (24)		
23	2	4	24	3	5	25	4	6	26	5	7	27	6	8
16	13	31	17	14	1	18	15	2	19	16	3	20	17	4
15	19	26	16	20	27	17	21	28	18	22	29	19	23	30
6	11	12	7	12	13	8	13	14	9	14	15	10	15	16
18	1	10	19	2	11	20	3	12	21	4	13	22	5	14
21	(29)	(2)	22	(30)	(3)	23	(31)	(4)	24	(1)	(5)	25	(2)	(6)
29	(8)	(13)	30	(9)	(14)	31	(10)	(15)	1	(11)	(16)	2	(12)	(17)
8	(5)	(19)	9	(6)	(20)	10	(7)	(21)	11	(8)	(22)	12	(9)	(23)
5	(9)	(11)	6	(10)	(12)	7	(11)	(13)	8	(12)	(14)	9	(13)	(15)
9	(14)	(1)	10	(15)	(2)	11	(16)	(3)	12	(17)	(4)	13	(18)	(5)
14	(28)	(4)	15	(29)	(5)	16	(30)	(6)	17	(31)	(7)	18	(1)	(8)
28	(30)	(31)	29	(31)	(1)	30	(1)	(2)	31	(2)	(3)	1	(3)	(4)
30	(17)	(26)	31	(18)	(27)	1	(19)	(28)	2	(20)	(29)	3	(21)	(30)
17	(24)	(12)	18	(25)	(13)	19	(26)	(14)	20	(27)	(15)	21	(28)	(16)
24	(25)	(10)	25	(26)	(11)	26	(27)	(12)	27	(28)	(13)	28	(29)	(14)
25	(3)	(23)	26	(4)	(24)	27	(5)	(25)	28	(6)	(26)	29	(7)	(27)
3	(22)	(16)	4	(23)	(17)	5	(24)	(18)	6	(25)	(19)	7	(26)	(20)
22	(7)	(15)	23	(8)	(16)	24	(9)	(17)	25	(10)	(18)	26	(11)	(19)
7	(27)	(6)	8	(28)	(7)	9	(29)	(8)	10	(30)	(9)	11	(31)	(10)
27	(21)	(18)	28	(22)	(19)	29	(23)	(20)	30	(24)	(21)	31	(25)	(22)

h 25 (25)			*h* 26 (26)			*h* 27 (27)			*h* 28 (28)			*h* 29 (29)			*h* 30 (30)		
8	7	9	29	8	10	30	9	11	31	10	12	1	11	13	2	12	14
1	18	5	22	19	6	23	20	7	24	21	8	25	22	9	26	23	10
0	24	31	21	25	1	22	26	2	23	27	3	24	28	4	25	29	5
1	16	17	12	17	18	13	18	19	14	19	20	15	20	21	16	21	22
3	6	15	24	7	16	25	8	17	26	9	18	27	10	19	28	11	20
6	(3)	(7)	27	(4)	(8)	28	(5)	(9)	29	(6)	(10)	30	(7)	(11)	31	(8)	(12)
3	(13)	(18)	4	(14)	(19)	5	(15)	(20)	6	(16)	(21)	7	(17)	(22)	8	(18)	(23)
3	(10)	(24)	14	(11)	(25)	15	(12)	(26)	16	(13)	(27)	17	(14)	(28)	18	(15)	(29)
0	(14)	(16)	11	(15)	(17)	12	(16)	(18)	13	(17)	(19)	14	(18)	(20)	15	(19)	(21)
4	(19)	(6)	15	(20)	(7)	16	(21)	(8)	17	(22)	(9)	18	(23)	(10)	19	(24)	(11)
9	(2)	(9)	20	(3)	(10)	21	(4)	(11)	22	(5)	(12)	23	(6)	(13)	24	(7)	(14)
2	(4)	(5)	3	(5)	(6)	4	(6)	(7)	5	(7)	(8)	6	(8)	(9)	7	(9)	(10)
4	(22)	(31)	5	(23)	(1)	6	(24)	(2)	7	(25)	(3)	8	(26)	(4)	9	(27)	(5)
2	(29)	(17)	23	(30)	(18)	24	(31)	(19)	25	(1)	(20)	26	(2)	(21)	27	(3)	(22)
9	(30)	(15)	30	(31)	(16)	31	(1)	(17)	1	(2)	(18)	2	(3)	(19)	3	(4)	(20)
0	(8)	(28)	31	(9)	(29)	1	(10)	(30)	2	(11)	(31)	3	(12)	(1)	4	(13)	(2)
8	(27)	(21)	9	(28)	(22)	10	(29)	(23)	11	(30)	(24)	12	(31)	(25)	13	(1)	(26)
7	(12)	(20)	28	(13)	(21)	29	(14)	(22)	30	(15)	(23)	31	(16)	(24)	1	(17)	(25)
2	(1)	(11)	13	(2)	(12)	14	(3)	(13)	15	(4)	(14)	16	(5)	(15)	17	(6)	(16)
1	(26)	(23)	2	(27)	(24)	3	(28)	(25)	4	(29)	(26)	5	(30)	(27)	6	(31)	(28)

It may be observed that if a different notation be required, a consecutive series of mbers, or any arbitrary set of symbols, or other designations, may be substituted the first arrangement, and the two collated together will supply a key for tting down all the others.

We may also remark that the same process will resolve the probl[illegible] generally for any other number, provided only that the modulus $m=6k$ [illegible] be a prime number, the number of school-boys being then $n = 2m+1$.

Extension of the Solution of Question 2261. By S. Bills.

Suppose 255 school boys to walk out for 127 days, 3 abreast, and let it [illegible] required to arrange them so that no two shall walk twice abreast.

Referring to the foregoing Solution of Question 2261, and employing M[illegible] Woolhouse's notation, I find 3 and 85 to be a pair of reciprocal roots of r^a t[illegible] modulus 127.

Taking $r=3$, the succession of remainders arising from dividing r^a by 12[illegible] while a takes every value from 1 to 126 will be in order as follows.

(Read this list downwards.)

3	22	119	26	106	100	56	72	20	62
9	66	103	78	64	46	41	89	60	59
27	71	55	107	65	11	123	13	53	50
81	86	38	67	68	33	115	39	32	23
116	4	114	74	77	99	91	117	96	69
94	12	88	95	104	43	19	97	34	80
28	36	10	31	58	2	57	37	102	113
84	108	30	93	47	6	44	111	52	85
125	70	90	25	14	18	5	79	29	1
121	83	16	75	42	54	15	110	87	
109	122	48	98	126	35	45	76	7	
73	112	17	40	124	105	8	101	21	
92	82	51	120	118	61	24	49	63	

Now if we take $\alpha=2$ and $\beta=7$; then $\alpha'=90$ and $\beta'=3$; and $r^{\alpha}=9$, $r^{\beta}=28$; and the necessary conditions, α and β unlike as to sign; and also α' and β' unlike as to sign will be satisfied, (see Solution of Quest. 2261).

Mr. Woolhouse's forms A and B, extended for 255 symbols, will be as follows:—

	A	
h	127	(127)
r	r^{43}	r^{85}
r^3	r^{45}	r^{87}
r^5	r^{47}	r^{89}
⋮	⋮	⋮
r^{41}	r^{83}	r^{125}
r^0	(r^{α})	(r^{β})
r^2	$(r^{\alpha+2})$	$(r^{\beta+2})$
r^4	$(r^{\alpha+4})$	$(r^{\beta+4})$
⋮	⋮	⋮
r^{124}	$(r^{\alpha+124})$	$(r^{\beta+124})$

	B	
h	127	(127)
r^0	r^{42}	r^{84}
r^2	r^{44}	r^{86}
r^4	r^{46}	r^{88}
r^6	r^{48}	r^{90}
⋮	⋮	⋮
r^{40}	r^{82}	r^{124}
r	$(r^{\alpha+1})$	$(r^{\beta+1})$
r^3	$(r^{\alpha+3})$	$(r^{\beta+3})$
r^5	$(r^{\alpha+5})$	$(r^{\beta+5})$
⋮	⋮	⋮
r^{125}	$(r^{\alpha+125})$	$(r^{\beta+125})$

Now taking $r=8$, $a=2$, and $\beta=7$; substituting in A, and remembering to take out the proper remainders from the above table, we find the primitive arrangement to be as follows. And by using the form B we should obtain another primitive arrangement. To save room however I shall not develop the subject further than to obtain one primitive arrangement.

h	127	(127)	1	(9)	(28)	31	(25)	(106)	72	(13)	(111)
			9	(81)	(125)	25	(98)	(65)	13	(117)	(110)
3	67	57	81	(94)	(109)	98	(120)	(77)	117	(37)	(101)
27	95	5	94	(84)	(92)	120	(64)	(58)	37	(79)	(20)
116	93	45	84	(121)	(66)	64	(68)	(14)	79	(76)	(53)
28	75	24	121	(73)	(86)	68	(104)	(126)	76	(49)	(96)
125	40	89	73	(22)	(12)	104	(47)	(118)	49	(60)	(102)
109	106	39	22	(71)	(108)	47	(42)	(46)	60	(32)	(29)
92	65	97	71	(4)	(83)	42	(124)	(33)	32	(34)	(7)
66	77	111	4	(36)	(112)	124	(100)	(43)	34	(52)	(63)
86	58	110	36	(70)	(119)	100	(11)	(6)	52	(87)	(59)
12	14	101	70	(122)	(55)	11	(99)	(54)	87	(21)	(23)
108	126	20	122	(82)	(114)	99	(2)	(105)	21	(62)	(80)
83	118	53	82	(103)	(10)	2	(18)	(56)	62	(50)	(85)
112	46	96	103	(38)	(90)	18	(35)	(123)	50	(69)	(3)
119	33	102	38	(88)	(48)	35	(61)	(91)	69	(113)	(27)
55	43	29	88	(30)	(51)	61	(41)	(57)	113	(1)	(116)
114	6	7	30	(16)	(78)	41	(115)	(5)			
10	54	63	16	(17)	(67)	115	(19)	(45)			
90	105	59	17	(26)	(95)	19	(44)	(24)			
48	56	23	26	(107)	(93)	44	(15)	(89)			
51	123	80	107	(74)	(75)	15	(8)	(39)			
78	91	85	74	(31)	(40)	8	(72)	(97)			

By passing this primitive arrangement through a cycle of 127 we should obtain the required arrangements for every day during the 127 days.

This question admits of numerous constructions, but it is unnecessary to develop it further as it necessarily leads to such lengthened results, and the method is the same for each.

The above arrangement may be relied upon as accurate.

NOTE ON THE FOUR-POINT PROBLEM. BY T. SAVAGE, M.A.

1. Let P, Q, R, S be four points taken at random. Then the chance that he terminated line PQ will intersect the terminated line RS is one-third of he chance that the four points are the angles of a convex quadrilateral. For if the quadrilateral be reentrant, they cannot intersect. If the quadrilateral be convex, they will intersect only when Q is the angle opposite to P, nd any one of the three Q, R, S may with equal probability be the opposite ngle.

2. Now the chance that the terminated lines PQ, RS will intersect is the ompound chance that P and Q lie on opposite sides of RS, and that R, S

lie on opposite sides of PQ. If then we may assume that infinite space is divided into two equal portions by any straight line, this chance is $\frac{1}{2} \times \frac{1}{2}$, and the chance of a convex quadrilateral is $\frac{3}{4}$.

3. The above assumption is undoubtedly true when the straight line passes through any point at a finite distance, but probably in that case only. The following appears to be free from this objection. If we examine four points which form a reentrant quadrilateral, we see that three of the six pairs of points lie on the same side of the line joining the other pair, but that in a convex quadrilateral four of the six pairs have this property. Hence if x be the chance that four points form a convex quadrilateral, the chance that two random points P, Q lie on the same side of the line joining two other random points R, S is $\frac{2}{3}x + \frac{1}{2}(1-x)$ or $\frac{1}{2}+\frac{1}{6}x$; and the chance that P, Q lie on opposite sides of RS is $\frac{1}{2}-\frac{1}{6}x$. Hence we must have $3(\frac{1}{2}-\frac{1}{6}x)^2 = x$, which gives us $x=9-6\sqrt{2}$.

4. The same result is obtained if we consider that the only ways in which we can obtain a convex quadrilateral are by supposing P, Q to be on the same or opposite sides of RS, according as R, S are on the same or opposite sides of PQ. This would give us the equation

$$(\tfrac{1}{2}+\tfrac{1}{6}x)^2+(\tfrac{1}{2}-\tfrac{1}{6}x)^2 = x,$$

which is the same as the former.

5. The chance that P, Q fall on opposite sides of a random line RS is then, it appears, $\sqrt{2}-1$. That this chance must be less than $\frac{1}{2}$ seems to be evident from the following. If any line divide space into two portions in the proportion of $x : y$, the chance that two random points lie on the same side of the line is $\frac{x^2+y^2}{(x+y)^2}$; that they lie on opposite sides it is $\frac{2xy}{(x+y)^2}$, and always less than the former, unless $x=y$. The above would appear to show that the average proportion of $x : y$ is $\sqrt{2}+1 : 1$.

6. In (2), (3), and (4) it is assumed that the two events,

P and Q on opposite sides of RS, and
R and S on opposite sides of PQ,

are independent. It is difficult completely to justify this, but it is evident that the probability of the second depends simply on the ratio of the two portions into which we consider space to be divided by the line PQ; and I do not see how the probable value of this can be affected by our knowledge that PQ joins two points taken at random, one on one side of the line RS and the other on the opposite side. As there is, however, doubtless a weak point here, I will proceed to show that the chance of a convex quadrilateral is less than $\frac{3}{5}$. Let λ be the chance that R, S are on opposite sides of PQ when P, Q are known to be on opposite sides of RS; and μ the same chance when P, Q are known to be on the same side of RS. Then, from the reasoning in (5), it is clear that neither λ nor μ can exceed $\frac{1}{2}$, and the modified forms of the equations in (3) and (4) are

$$3(\tfrac{1}{2}-\tfrac{1}{6}x)\lambda = x, \quad \text{and} \quad (\tfrac{1}{2}-\tfrac{1}{6}x)\lambda+(\tfrac{1}{2}+\tfrac{1}{6}x)(1-u) = x;$$

from which, by eliminating x, we find that

$$(\lambda+1)(\mu+1) = 2,$$

which gives us $\frac{1}{3}$ as an inferior limit to the value of either λ or μ. If we express x in terms of (1) λ or (2) μ, we find

$$x = \frac{3\lambda}{2+\lambda}, \text{ or } = \frac{3(1-\mu)}{3+\mu}.$$

By substituting either for λ its greatest or for μ its least value, we find the true value of x must be less than $\frac{3}{5}$.

7. It is curious to observe that just as we obtain the maximum value of λ or μ by supposing space to be divided into two equal portions by any straight line, so we appear to obtain the minimum value by assuming that the intersection of two random lines is a random point. For let PQ, RS meet in O. Then if O be a random point, of the three O, P, Q any one may with equal probability be the middle point, and therefore the chance that O is the middle point is $\frac{1}{3}$. The discordance between the results obtained by the methods of § 3 and § 4, when we make either of these assumptions, appears to be a sufficient proof of their falsity.

1971. (Proposed by C. W. MERRIFIELD, F.R.S.)—In a given tetrahedron to find a set of rectangular coordinate axes, such that each axis shall pass through a pair of opposite edges.

Quaternion Solution by T.

Let α, β, γ be three (vector) edges of the tetrahedron, one corner being the origin. Let ρ be the vector of the origin of the sought rectangular system, which may be called i, j, k (unknown vectors.) The condition that i, drawn from ρ, intersects α is

$$S.i\alpha\rho = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

That it intersects the opposite edge, whose equation is $\varpi = \beta + x\,(\beta-\gamma)$, the condition is

$$S.i(\beta-\gamma)\,(\rho-\beta) = 0, \text{ or } Si\left\{(\beta-\gamma)\rho-\beta\gamma\right\} = 0 \;\ldots\ldots (2).$$

There are two other equations like (1), and two like (2), which can be at once written down.

Put $\beta-\gamma=\alpha_1$, $\gamma-\alpha=\beta_1$, $\alpha-\beta=\gamma_1$, $V\beta\gamma=\alpha_2$, $V\gamma\alpha=\beta_2$, $V\alpha\beta=\gamma_2$, $V\alpha_1\alpha = \alpha_3$, $V\beta_1\beta = \beta_3$, $V\gamma_1\gamma = \gamma_3$;

and the six become

$$\begin{aligned} S.i\alpha\rho &= 0, & S.i\alpha_1\rho - Si\alpha_2 &= 0,\\ S.j\beta\rho &= 0, & S.j\beta_1\rho - Sj\beta_2 &= 0,\\ S.k\gamma\rho &= 0, & S.k\gamma_1\rho - Sk\gamma_2 &= 0. \end{aligned}$$

The two in i give $\quad i \parallel \alpha S\alpha_2\rho - \rho\,(S\alpha\alpha_2 + S\alpha_3\rho)$.

Similarly, $\quad j \parallel \beta S\beta_2\rho - \rho\,(S\beta\beta_2 + S\beta_3\rho)$, and $k \parallel \gamma S\gamma_2\rho - \rho\,(S\gamma\gamma_2 + S\gamma_3\rho)$.

The conditions of rectangularity, viz.,

$$Sij = 0, \quad Sjk = 0, \quad Ski = 0,$$

at once give three equations of the fourth order, the first of which is

$$\begin{aligned} 0 = S\alpha\beta\, S\alpha_2\rho\, S\beta_2\rho - S\alpha\rho\, S\alpha_2\rho\,(S\beta\beta_2 + S\beta_3\rho) - S\beta\rho\, S\beta_2\rho\,(S\alpha\alpha_2 + S\alpha_3\rho)\\ + \rho^2\,(S\alpha\alpha_2 + S\alpha_3\rho)\,(S\beta\beta_2 + S\beta_3\rho). \end{aligned}$$

The required origin of the rectangular system is thus given as the intersection of three surfaces of the fourth order. How many such points of intersection there may be, I have not endeavoured to find.

2466. (Proposed by H. MURPHY.)—If four points A, B, C, D be either in the same plane or not, and if the three rectangles AB . CD, AC . DB, AD . BC be taken; the sum of any two of them is greater than the third, except when the points lie on the circumference of a circle.

Solution by PROFESSOR CAYLEY.

Write for shortness BC $= f$, CA $= g$, AB $= h$; AD $= a$, BD $= b$, CD $= c$; then, Lemma, if r be the radius of the sphere circumscribed about the tetrahedron ABCD, we have

$$4r^2 \left\{ \begin{array}{l} -a^4f^2-b^4g^2-c^4h^2-f^2g^2h^2 \\ +(a^2f^2+b^2c^2)(g^2+h^2-f^2) \\ +(b^2g^2+c^2a^2)(h^2+f^2-g^2) \\ +(c^2h^2+a^2b^2)(f^2+g^2-h^2) \end{array} \right\} = \begin{array}{l} 2b^2c^2g^2h^2+2c^2a^2h^2f^2+2a^2b^2f^2g^2 \\ \quad -a^4f^4-b^4g^4-c^4h^4, \end{array}$$

where the left-hand side is $=576\,\mathrm{V}^2r^2$, if V be the volume of the tetrahedron.

Suppose first that the points are not in the same plane, then the left-hand side ($=576\,\mathrm{V}^2r^2$) is positive; therefore the right-hand side is also positive, or putting for shortness, $af = \alpha$, $bg = \beta$, $ch = \gamma$, we have

$2\beta^2\gamma^2+2\gamma^2\alpha^2+2\alpha^2\beta^2-\alpha^4-\beta^4-\gamma^4 = +$, that is, $4\beta^2\gamma^2-(\alpha^2-\beta^2-\gamma^2)^2 = +$, and thence $\alpha<\beta+\gamma$; for if α were equal to or greater than $\beta+\gamma$, say $\alpha = \beta+\gamma+x$, the left-hand side would be $4\beta^2\gamma^2-\left\{2\beta\gamma+2(\beta+\gamma)x+x^2\right\}^2$, which vanishes if $x = 0$, and is negative for x positive. Similarly $\beta<\gamma+\alpha$, $\gamma<\alpha+\beta$; and the theorem is thus proved for the case where the four points are not in a plane.

Starting from this general case, if we imagine the point D continually to approach and ultimately to coincide with the plane ABC, but so as not to be in the circle ABC, then the expression $2\beta^2\gamma^2+2\gamma^2\alpha^2+2\alpha^2\beta^2-\alpha^4-\beta^4-\gamma^4$, which does not vanish in the limit, is throughout equal to the positive quantity $576\,\mathrm{V}^2r^2$ (in the limit V is $= 0$ and $r = \infty$, but $\mathrm{V}r$ is finite, and of course V^2r^2 is positive), that is, the expression in question is $= +$, and the theorem follows as before. Of course when the four points are in a circle, then the expression is $= 0$, and consequently one of the quantities α, β, γ is equal to the sum of the other two.

The lemma is at once proved by means of my theorem for the relation between the distances of five points in space, [*Cambridge Mathematical Journal*, Vol. II. (1841) p. 269,] viz., if the point 1 is the centre of the circumscribed sphere, and the points 2, 3, 4, 5, are the points A, B, C, D respectively, then the relation in question, viz.,

$$\left| \begin{array}{cccccc} 0\ , & (12)^2, & (13)^2, & (14)^2, & (15)^2, & 1 \\ (21)^2, & 0\ , & (23)^2, & (24)^2, & (25)^2, & 1 \\ (31)^2, & (32)^2, & 0\ , & (34)^2, & (35)^2, & 1 \\ (41)^2, & (42)^2, & (43)^2, & 0\ , & (45)^2, & 1 \\ (51)^2, & (52)^2, & (53)^2, & (54)^2, & 0\ , & 1 \\ 1, & 1, & 1\ , & 1\ , & 1\ , & 0 \end{array} \right| = 0$$

becomes
$$\begin{vmatrix} 0, & r^2, & r^2, & r^2, & r^2, & 1 \\ r^2, & 0, & h^2, & g^2, & a^2, & 1 \\ r^2, & h^2, & 0, & f^2, & b^2, & 1 \\ r^2, & g^2, & f^2, & 0, & c^2, & 1 \\ r^2, & a^2, & b^2, & c^2, & 0, & 1 \\ 1, & 1, & 1, & 1, & 1, & 0 \end{vmatrix} = 0.$$

Multiplying the last line by $-r^2$ and adding it to the first line, this is
$$\begin{vmatrix} -r^2, & 0, & 0, & 0, & 0, & 1 \\ r^2, & 0, & h^2, & g^2, & a^2, & 1 \\ r^2, & h^2, & 0, & f^2, & b^2, & 1 \\ r^2, & g^2, & f^2, & 0, & c^2, & 1 \\ r^2, & a^2, & b^2, & c^2, & 0, & 1 \\ 1, & 1, & 1, & 1, & 1, & 0 \end{vmatrix} = 0,$$

and then proceeding in the same way with the first and last columns, the equation is
$$\begin{vmatrix} -2r^2, & 0, & 0, & 0, & 0, & 1 \\ 0, & 0, & h^2, & g^2, & a^2, & 1 \\ 0, & h^2, & 0, & f^2, & b^2, & 1 \\ 0, & g^2, & f^2, & 0, & c^2, & 1 \\ 0, & a^2, & b^2, & c^2, & 0, & 1 \\ 1, & 1, & 1, & 1, & 1, & 0 \end{vmatrix} = 0,$$

which is in fact the equation of the Lemma. See my papers in the *Quarterly Journal of Mathematics*, Vol. III. (1859), pp. 275—277, and Vol. V. (1861), pp. 381—384.

COR.—It appears by the demonstration that for any four points not in the same plane, the expression
$$-a^4f^2-b^4g^2-c^4h^2-f^2g^2h^2+(a^2f^2+b^2c^2)(g^2+h^2-f^2)$$
$$+(b^2g^2+c^2a^2)(h^2+f^2-g^2)+(c^2h^2+a^2b^2)(f^2+g^2-h^2)$$
is always positive.

1496. (Proposed by MATTHEW COLLINS, B.A.)—Prove that a triangular pyramid whose vertices are A, B, C, D, and a parallelepiped formed from it as follows, have the same centre of gravity; viz., through any point in each of the opposite edges AB and CD draw straight lines parallel to the other edge, we thus get two parallel plane faces of the parallelepiped; two other parallel faces of it are similarly obtained from the opposite edges AC and BD; and the third pair of faces are obtained from the remaining two opposite edges AD and BC.

Solution by W. S. B. WOOLHOUSE, F.R.A.S.

Each pair of parallel planes determined as described in the Question are those which pass through two opposite edges of the tetrahedron, and the line of shortest distance between the edges is normal to the planes. The parallelism of the planes proves the figure to be a parallelepiped. Now, if with respect to the parallelepiped, planes be drawn through the centre of gravity parallel to each pair of opposite faces, they will manifestly bisect every line which connects those faces, and therefore they will respectively bisect the three lines which join the middle points of the opposite edges of the tetrahedron which lie in those faces. But the three last mentioned lines are known to mutually bisect one another in the centre of gravity of the tetrahedron. Therefore the tetrahedron and parallelepiped have the same centre of gravity.

2459. (Proposed by R. BALL, M.A.)—If ω be a cube root of unity, express the roots of the equation

$$ax^3+3bx^2+\frac{3}{a}\left\{b^2+\omega\,(ac-b^2)\right\}x+\frac{1}{a^2}\left\{a^2d+3\,(\omega-1)\,(ab-b^3)\right\}=0,$$

in terms of α, β, γ the roots of $ax^3+3bx^2+3cx+d=0$.

Solution by the REV. ROBERT HARLEY, F.R.S.

Writing in the first cubic X, and in the second X′, in place of $ax+b$, we have

$$X^3+3\omega\,(ac-b^2)\,X+a^2d-3abc+2b^3=0 \;\ldots\ldots\ldots\ldots (1),$$

$$X'^3+3\,(ac-b^2)\,X'+a^2d-3abc+2b^3=0 \;\ldots\ldots\ldots\ldots (2).$$

Now (2) − (1) gives $X'^3-X^3+3\,(ac-b^2)\,(X'-\omega X)=0$;

or $$(X'-\omega X)\left\{X'^2-\omega XX'+\omega^2X^2+3\,(ac-b^2)\right\}=0;$$

whence, either $X=\omega^2X'$,

or $$X=\tfrac{1}{2}\omega^2\left\{X'+\surd(-3)\,\surd[X'^2+4\,(ac-b^2)]\right\}.$$

The latter result is incongruous with the condition $\Sigma X=0$, and must therefore be rejected. The former gives

$$x=\omega^2x'-b\,(1-\omega\;),$$

where $x'=\alpha$, β or γ.

1805. (Proposed by Professor SYLVESTER.)—Prove that in general

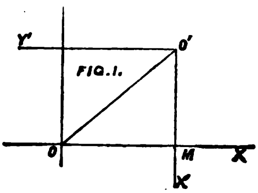
$$\int_0^l dx\int_0^x dy\,F(x,y)=\int_0^l dx\int_0^x dy\,F(l-y,\,l-x)\ldots\ldots\ldots(1),$$

and hence show instantaneously that

$$\int_0^\pi d\theta\int_0^\theta d\theta'\,(\sin\theta\sin\theta')^{2i-1}\sin(\theta-\theta')=\frac{1.3.5.\ldots.(4i-1)}{2.46.\ldots.4i}\,\frac{\pi}{i}\ldots(3);$$

and (3) apply the latter integral to prove that the chance of four points taken at random in a circle forming the apices of a reentrant quadrilateral is $\frac{35}{12\pi^2}$.

Solution by THOMAS SAVAGE, M.A.



1. The first of the above integrals is the volume included between the surface $z = F(x, y)$, the plane of xy, and the three planes perpendicular to the plane of xy, and having as their traces on this plane the three lines $y = 0$, $y = x$, $x = l$. If we transfer the origin (Fig. 1) to a point (O′) whose coordinates are $(l, l, 0)$, and take for our new axes of x and y two lines (O′X′, O′Y′) parallel respectively to (OY, OX) but drawn in the opposite direction, it is clear that the equation to the surface becomes $z = F(l-y, l-x)$, and that to find the same volume as before, the limits of the integration with respect to y and x will be $(0, x)$, $(0, l)$ respectively. Hence the truth of the first equation is manifest.

2. A particular case of the above general formula is evidently

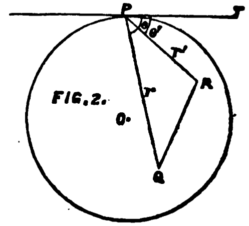
$$\int_0^\pi d\theta \int_0^\theta d\theta' (\sin\theta \sin\theta')^{2i-1} \cos\theta \sin\theta'$$

$$= -\int_0^\pi d\theta \int_0^\theta d\theta' (\sin\theta \sin\theta')^{2i-1} \sin\theta \cos\theta' ;$$

$$\therefore \int_0^\pi d\theta \int_0^\theta d\theta' (\sin\theta \sin\theta')^{2i-1} \sin(\theta-\theta')$$

$$= 2\int_0^\pi d\theta \int_0^\theta d\theta' (\sin\theta \sin\theta')^{2i-1} \sin\theta \cos\theta'$$

$$= \frac{1}{i}\int_0^\pi (\sin\theta)^{4i} d\theta = \frac{1.3.5.\ldots.(4i-1)}{2.4.6.\ldots.4i} \cdot \frac{\pi}{i}.$$



3. Let P, Q, R, S be four points taken at random within a circle whose centre is O. One of these four points (P suppose) must be more distant from the centre than the others. Then Q, R, S are three points taken at random within the circle whose radius is OP, and the chance that PQRS should be the apices of a reentrant quadrilateral is the chance that one of the three points Q, R, S should fall within the triangle formed by the other two and P, or is three times the average area of the triangle PQR divided by the area of the circle whose radius is OP. Referring the circle to polar coordinates, P being the pole, and the tangent at P the initial line, the average area of PQR is easily seen to be

$$\frac{\iiiint \frac{1}{2} rr' \sin(\theta-\theta') . r dr d\theta . r' dr' d\theta'}{\iiiint r dr d\theta . r' dr' d\theta'}$$

the limits of r, r', θ', θ being respectively

$(0, 2\text{OP} \sin\theta)$, $(0, 2\text{OP} \sin\theta')$, $(0, \theta)$, $(0, \pi)$.

Hence the average area of the triangle PQR is

$$\frac{\frac{1}{2}(\mathrm{OP})^6}{\frac{1}{2}\pi^2(\mathrm{OP})^4}\int_0^\pi d\theta\int_0^\theta d\theta'\,.\,\frac{64}{9}(\sin\theta\sin\theta')^3\sin(\theta-\theta')$$

$$=\frac{64(\mathrm{OP})^2}{9\pi^2}\cdot\frac{1\,.\,3\,.\,5\,.\,7}{2\,.\,4\,.\,6\,.\,8}\cdot\frac{\pi}{2}=\frac{35(\mathrm{OP})^2}{36\pi};$$

and the chance of a reentrant quadrilateral is

$$\frac{3}{\pi(\mathrm{OP})^2}\cdot\frac{35(\mathrm{OP})^2}{36\pi}=\frac{35}{12\pi^2}.$$

2356. (Proposed by Captain Clarke, R.E., F.R.S.)—**A tetrahedron is thrown into the air and a small shot fired through it; show that the chance of one of the faces (which is marked) being perforated, is 2F ÷ S, where F is the area of the marked face and S the whole surface of the tetrahedron.**

Solution by W. S. B. Woolhouse, F.R.A.S.

The relation stated in the Question may be proved to be true generally for any polyhedron. If we conceive a large number of shots to be fixed through the polyhedron, and suppose the polyhedron to occupy every possible position with respect to the direction in which the shots are thrown, a little consideration will show that the number of shots which perforate two faces whose areas are F, F′ will be proportional to those areas. For, in integrating for the various positions of the plane of F it is evident that the plane of F′ will equally require to be integrated through the same positions; and as the orthographic projections are proportional to the areas, it is hence manifest that the resulting number of perforations must be also proportional to those areas. Hence if only one perforation were made, the chance of its being in the face F would be F ÷ S. But as two distinct faces are perforated by the shot, one in entering and another in quitting the polyhedron, the chance of the assigned face F being one of them is 2F ÷ S.

1843. (Proposed by the Editor.) — **1. Three points being taken at random within a circle, find the chance that the circle drawn through them will lie wholly within the given circle.**

2. Four points being taken at random within a sphere, find the chance that the sphere drawn through them will lie wholly within the given sphere.

Solution by Samuel Roberts, M.A.

1. The centre of a circle through three favourable points must of course lie within the given circle, and any point within this circle may be taken as

the centre of a system of interior circles. Suppose we take any such point P, c being its distance from the centre, the radius of a circle about P must not exceed $r-c$, r being the radius of the given circle.

Now if a circle be drawn about P as centre, with the radius t, the points on its circumference can be combined together three and three in $\frac{4}{3}\pi^3 t^3$ ways, and $\int_0^{r-c} \frac{4}{3}\pi^3 t^3 dt$ gives the total number of such combinations relative to P. The probability required in this case is therefore

$$p = \left(\tfrac{8}{3}\pi^4 \int_0^r \int_0^{r-c} cde \,.\, t^3 dt\right) \div \left(\frac{\pi^3 r^6}{6}\right) = \frac{4\pi}{r^6}\int_0^r (r-c)^4\, cdc = \frac{2\pi}{15}.$$

2. Adopting a similar notation for the sphere, the number of favourable combinations relative to the point P is $\int_0^{r-c} \frac{32}{3}\pi^4 t^8 dt$, and consequently the probability required in this case is

$$p = \left(\tfrac{128}{3}\pi^5 \int_0^r \int_0^{r-c} c^2 dc \,.\, t^8 dt\right) \div \left(\frac{32\pi^4 r^{12}}{243}\right) = \frac{36\pi}{r^{12}}\int_0^r (r-c)^9\, c^2 dc = \frac{3\pi}{55}.$$

[Mr. ROBERTS remarks that the problem may be readily extended to hyper-space of n dimensions; then, the content of the hyper-sphere being

$$\frac{\pi^{\frac{1}{2}n} r^n}{\Gamma(\frac{1}{2}n+1)}, \text{ the result is } \frac{n^n \pi^{\frac{1}{2}n}}{r^{n(n+1)}\,\Gamma(\frac{1}{2}n+1)}\int_0^r (r-c)^{n^2} c^{n-1} dc,$$

which includes all the cases for integral values of n.]

II. *Solution by* W. S. B. WOOLHOUSE, F.R.A.S.

1. The required chance being obviously the same for all circles, will remain unaltered if the given circle be augmented by a concentric annulus; and it will therefore follow that the same probability must result if the new cases thence arising be treated separately. It will be observed that the investigation is affected by this peculiarity, that the new cases which satisfy the proposed condition are principally derived from a conversion of a portion of the old combinations which were previously unfavourable. Thus the newly acquired favourable cases are those in which the circle passing through any three points passes into the annulus without going beyond it; whereas the total new cases are those in which one, or more, of the three points is situated in the annulus. If we now conceive the annulus to be diminished without limit, the total new cases will ultimately become those which have one of the points in the periphery of the given circle, and the additional favourable cases will be those in which the circle drawn through the three points is tangential to the given circle. To estimate these favourable cases, let R denote the radius of the given circle, and ρ that of the circle described through the three points; then, as the points may be anywhere in the circumference, the positions are $(2\pi\rho)^3$. Also, as the circles are in contact, the positions of

the centre, from ρ to $\rho+d\rho$, are $2\pi(\mathrm{R}-\rho)\,d\rho$. Therefore the estimated number of favourable positions is

$$(2\pi)^4\int\rho^3d\rho\,(\mathrm{R}-\rho)=(2\pi)^4\,\mathrm{R}^5\left(\tfrac{1}{4}-\tfrac{1}{5}\right)=\tfrac{4}{5}\pi^4\,\mathrm{R}^5.$$

The total number of new cases, bearing in mind that each of the three points may be in the periphery and the remaining two on the surface, is $3\,(2\pi\mathrm{R})\,.\,(\pi\mathrm{R}^2)^2=6\pi^3\mathrm{R}^5$. Hence by division the required probability is $\dfrac{2\pi}{15}$.

2. If four points be taken at random within a sphere, the chance that the sphere drawn through them shall be wholly within the given sphere may be obtained in a similar manner.

Here the estimated number of acquired favourable positions is

$$\int(4\pi\rho^2)^4\,.\,4\pi\,(\mathrm{R}-\rho)^2\,d\rho=(4\pi)^5\int\rho^8d\rho\,(\mathrm{R}^2-2\mathrm{R}\rho+\rho^2)$$
$$=(4\pi)^5\,\mathrm{R}^{11}\left(\frac{1}{9}-\frac{2}{10}+\frac{1}{11}\right)=\frac{(4\pi)^5}{495}\,\mathrm{R}^{11}.$$

And the total number of new cases is

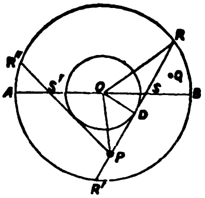
$$4\,(4\pi\mathrm{R}^2)\,.\left(\frac{4}{3}\pi\mathrm{R}^3\right)^3=\frac{4^5\pi^4}{3^3}\,\mathrm{R}^{11}.$$

Therefore the probability of the sphere being included within the given one is $\dfrac{3\pi}{55}$.

N.B.—This solution is only approximative, the estimated points not being in strictness equally distributed. An accurate solution of the question would be very complicated.

1849. (Proposed by Professor SYLVESTER.)—Two points are taken at random, one on each side of a given diameter of a circle; what is the probability that the chord drawn through them shall not exceed a given length?

Solution by STEPHEN WATSON.



Let AOB be the diameter, RR′ a chord of the given length, touching a concentric circle in D, cutting AB in S, and passing through P, one of the random points. Put $\mathrm{OB}=a$, $\mathrm{OP}=x$, $\angle\mathrm{ROD}=\alpha$, $\angle\mathrm{POB}=\phi$, $\angle\mathrm{DOP}=\theta$; then the chord through PQ will be less than RR′ so long as Q lies within either of the spaces BSR, AS′R″, the area of the first of which is

$$\mathrm{BSR}=\tfrac{1}{2}a^2\left\{\alpha+\theta-\phi+\cos^2\alpha\tan(\phi-\theta)-\sin\alpha\cos\alpha\right\}\;\ldots\ldots\ldots\ldots(1).$$

Now the result obtained by taking BSR alone must be doubled, because AS′R″ obviously passes through the same variations of area; also the limits of ϕ are from 0 to $\alpha+\theta$, of x from $a\cos\alpha$ to a, and (since $x\cos\theta=a\cos\alpha$) of θ from 0 to α; moreover an element of the circle at P is $xdxd\phi$, and the total number of positions of P and Q is expressed by $\frac{1}{4}a^4\pi^2$; hence the required chance is

$$p = \frac{8}{a^4\pi^2}\int_{a\cos\alpha}^{a}\int_{0}^{\alpha+\theta}(1)\,x\,dx\,d\phi$$

$$= \frac{2}{a^2\pi^2}\int_{a\cos\alpha}^{a}\left\{(\alpha+\theta)^2-(\alpha+\theta)\sin 2\alpha+2\cos^2\alpha\log\left(\frac{\cos\theta}{\cos\alpha}\right)\right\}x\,dx \quad \ldots\ldots (2)$$

Now remembering that $x\cos\theta = a\cos\alpha$, we have

$$\int x\,dx = \tfrac{1}{2}x^2,\quad \int \theta\, x\,dx = \tfrac{1}{2}\left(\theta x^2 - a^2\cos^2\alpha\tan\theta\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (3, 4),$$

$$\int \theta^2 x\,dx = \tfrac{1}{2}x^2\theta^2 - a^2\cos^2\alpha\,(\theta\tan\theta + \log\cos\theta) \quad \ldots\ldots\ldots\ldots\ldots\ldots (5),$$

$$\int \log\left(\frac{\cos\theta}{\cos\alpha}\right)x\,dx = \int \log\left(\frac{a}{x}\right)x\,dx = \tfrac{1}{2}x^2\log\left(\frac{a}{x}\right)+\tfrac{1}{4}x^2 \quad \ldots\ldots\ldots\ldots (6).$$

Taking (3), (4), (5), (6) between the above stated limits of x and θ, and substituting the results in (2), it becomes

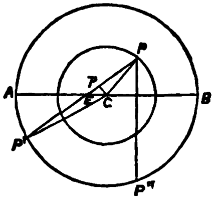
$$p = \frac{1}{\pi^2}\left\{\alpha\,(\alpha-\sin 2\alpha)\,(3+\sin^2\alpha)+\sin^2\alpha\cos^2\alpha\,(3-2\log\cos\alpha)\right\}.$$

[Putting, in Mr. Watson's general result, $\alpha = \frac{1}{6}\pi$, we find that the chance of the chord being not greater than the *radius* is

$$p = \frac{13}{144} - \frac{13\sqrt{3}}{48\pi} + \frac{3}{16\pi^2}\left(3+\log\frac{4}{3}\right),\ \text{or } p = \cdot 00342, \text{ nearly.}$$

The value of the analogous chance when the two points are taken at random *anywhere within the circle* is ·0117, as shown in Professor SYLVESTER's solution of Question 1818, on p. 20 of Vol. V. of the *Reprint*.]

II. *Solution by* SAMUEL ROBERTS, M.A.



Let AB be the given diameter of a circle whose radius is unity. Describe an inner concentric circle, and let P be a point on its upper semi-circumference. Through P there can be drawn in the original circle two chords of a given length. Let PP′ be one of these chords, meeting the diameter at E, which must by the conditions of the question lie between the centre and A or B. For the particular position of P, AEP′ is the favourable area outside PP′. Now it may be inferred from the symmetry of the figure that the favourable areas outside PP″, the other chord, will be the same in inverted order as those outside PP′, so that we need only double the result relative to PP′ alone.

Draw Cp perpendicular to PP′, and let $\angle$ PCB $= \phi$, $\angle$ PC$p = \theta$, $\angle$ P′C$p = \alpha$; then we have $\angle$ ACP′ $= \phi+\alpha+\theta-\pi$, $\angle$ CEP $= \phi+\theta-\frac{1}{2}\pi$;

therefore area ACP′ $= \dfrac{\phi+\alpha+\theta-\pi}{2} - \dfrac{\cos\alpha\sin(\phi+\alpha+\theta)}{2\cos(\phi+\theta)}$.

This expression is to be multiplied by $\rho d\rho d\phi$ and integrated as to ϕ between the limits $(\pi-\alpha-\theta, \pi)$ and as to ρ between the limits $(\cos\alpha, 1)$.

The first integration gives

$$\left\{\frac{(\alpha+\theta)^2}{4}-\frac{(\alpha+\theta)\cos\alpha\sin\alpha}{2}+\frac{\cos^2\alpha}{2}(\log\cos\theta-\log\cos\alpha)\right\}\rho d\rho.$$

Substituting for $\rho d\rho$ its equivalent $\dfrac{\cos^2\alpha\sin\theta\, d\theta}{\cos^3\theta}$, we have to integrate

$$\cos^2\alpha\int_0^\alpha\left\{\frac{\theta^2}{4}+\frac{\alpha-\cos\alpha\sin\alpha}{2}\theta+\frac{\alpha^2-2\alpha\cos\alpha\sin\alpha-2\cos^2\alpha\log\cos\alpha}{4}\right.$$
$$\left.+\frac{\cos^2\alpha\log\cos\theta}{2}\right\}\frac{\sin\theta\, d\theta}{\cos^3\theta}$$

But $$\int\frac{\theta^2\sin\theta\, d\theta}{\cos^3\theta}=\frac{1}{2}\left(\frac{\theta^2}{\cos^2\theta}-2\theta\tan\theta-2\log\cos\theta\right),$$

$$\int\frac{\theta\sin\theta\, d\theta}{\cos^3\theta}=\frac{1}{2}\left(\frac{\theta}{\cos^2\theta}-\tan\theta\right),$$

$$\int\frac{\sin\theta\, d\theta}{\cos^3\theta}=\frac{1}{2\cos^2\theta},\quad\int\frac{\sin\theta\log\cos\theta\, d\theta}{\cos^3\theta}=\frac{1}{2}\left(\frac{\log\cos\theta}{\cos^2\theta}+\frac{1}{2\cos^2\theta}\right).$$

Taking these integrals between the specified limits, we get

$$\tfrac{1}{8}\left\{\alpha(\alpha-\sin 2\alpha)(3+\sin^2\alpha)+\sin^2\alpha\cos^2\alpha(3-2\log\cos\alpha)\right\}.$$

This must be multiplied by 2 and divided by $\frac{1}{4}\pi^2$, giving for the required probability the same value as in the foregoing solution.

2340. (Proposed by R. TUCKER, M.A.) - The circles of curvature at the extremities of a pair of semi-conjugate diameters of an ellipse are drawn; find (1) the envelop of their radical axis and (2) the curve upon which their points of intersection lie.

Solution by the PROPOSER; *the* REV. J. L. KITCHIN, M.A.; *and others.*

1. Let θ be the eccentric angle of the extremity of one diameter, then $x = a\cos\theta$ and $y = b\sin\theta$.

Now the equation to the circle of curvature is

$$(X-\alpha)^2+(Y-\beta)^2=\rho^2\dots\dots\dots\dots(1),$$

but $\alpha = ae^2\cos^3\theta$, $\beta=-\dfrac{a^2e^2}{b}\sin^3\theta$, and $\rho=-\dfrac{(a^2\sin^2\theta+b^2\cos^2\theta)^{\frac{3}{2}}}{ab}$,

hence the equation (1) becomes

$$X^2+Y^2-2ae^2\cos^3\theta\,.\,X+\frac{2a^2e^2}{b}\sin^3\theta\,.\,Y=3a^2e^2\sin^2\theta+2b^2-a^2\dots\dots(2).$$

The equation to the other circle will be obtained by changing θ into $\frac{1}{2}\pi+\theta$, whence we obtain

$$X^2+Y^2+2ae^2\sin^3\theta\,.\,X+\frac{2a^2e^2}{b}\cos^3\theta\,.\,Y=3a^2e^2\cos^2\theta+2b^2-a^2\quad\ldots(3).$$

The equation to radical axis of (2) and (3) is

$$bX\,(\cos^3\theta+\sin^3\theta)+aY\,(\cos^3\theta-\sin^3\theta)=\tfrac{3}{2}ab\cos 2\theta\,\ldots\ldots\ldots\ldots(4).$$

Differentiating, we have

$$bX\,(\cos\theta-\sin\theta)+aY\,(\cos\theta+\sin\theta)=2ab\,\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Put $\cos\theta-\sin\theta=\lambda$ and $\cos\theta+\sin\theta=\mu$; then (4) and (5) become

$$\mu bX\,(1+\lambda^2)+\lambda aY\,(1+\mu^2)=3\lambda\mu ab,\quad \lambda bX+\mu aY=2ab\ldots\,(6,7);$$

that is, making use of (7) in (6), the equation of the envelop is obtained by eliminating λ and μ between the three equations

$$\frac{bX}{\lambda}+\frac{aY}{\mu}=ab,\quad \lambda bX+\mu aY=2ab,\quad \lambda^2+\mu^2=2\ldots\ldots(8,9,10).$$

Multiplying (8) by (10) and equating the result to (9), we get

$$-\frac{\lambda^3}{\mu^3}=\frac{bX}{aY},$$

hence we find that the curve required has for its equation

$$\left\{\pm(bX)^{\frac{2}{3}}\mp(aY)^{\frac{2}{3}}\right\}\left\{(bX)^{\frac{2}{3}}+(aY)^{\frac{2}{3}}\right\}^{\frac{1}{2}}=2^{\frac{1}{2}}ab.$$

2. To find the curve on which the points of intersection lie, we have to eliminate λ and μ between the two equations

$$ae^2X\lambda\,(1+\mu^2)-\frac{a^2e^2}{b}Y\mu\,(1+\lambda^2)=2\,(X^2+Y^2)-(a^2+b^2),$$

$$\mu b\,(1+\lambda^2)X+a\lambda\,(1+\mu^2)\,Y=3ab\lambda\mu,\quad \lambda^2+\mu^2=2,$$

the first equation being obtained by adding (2) and (3) together; that is, we have to eliminate λ and μ between three equations of the form

$$A\lambda\,(1+\mu^2)+B\mu\,(1+\lambda^2)=c,\quad A'\mu(1+\lambda^2)+B'\lambda(1+\mu^2)=c'\lambda\mu,\quad \lambda^2+\mu^2=2;$$

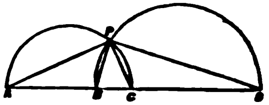
that is to say, between three equations of the form

$$\kappa\mu\,(1+\lambda^2)=cB'-Ac'\lambda\mu,\quad -\kappa\lambda\,(1+\mu^2)=cA'-Bc'\lambda\mu,\quad \lambda^2+\mu^2=2\,..\,(11,12,13).$$

Squaring and adding (11) and (12) with the help of (13) we get a quadratic for the determination of $\lambda\mu$; hence λ, μ are known, and the curve can be determined.

2386. (Proposed by G. O. HANLON.)—If the side AD of a square be divided harmonically in B and C, and squares be placed on AB, BC, CD, falling without the square on AD, prove that the circles round the four squares pass through a common point.

Solutions by J. Dale; *the* Proposer; R. Tucker, M.A.; *and others.*



1. Describe semicircles on AC, BD cutting in P, and join PB, PC; then as (P . ABCD) is a harmonic pencil, of which the conjugate rays PA, PC, and PB, PD are at right angles, therefore each of the angles APB, BPC, CPD is half a right angle; and circles described on the segments AB, BC, CD on the side towards P, and on AD on the side opposite P, each containing half a right angle, will pass through P; and these circles will circumscribe the squares placed on AB, BC, CD, AD.

2. *Otherwise:* let ADEF be the square on AD; then FB and EC meet on the circle round ADEF, say in the point P. Then FPA, FPE, EPD are evidently each half a right angle, from which the proof immediately follows (that is, the circles all pass through P).

3. *Or again:* let AB = h, AC = k, AD = a; then, taking A as the origin of rectangular coordinates, the given line being the axis of x, the equations to the circles circumscribing the squares placed on AB, BC, CD, and AD towards the opposite side, are

$$x^2+y^2 = hx+hy \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$
$$x^2+y^2 = (k+h)\,x+(k-h)\,y-hk \quad\ldots\ldots\ldots\ldots (2),$$
$$x^2+y^2 = (a+k)\,x+(a-k)\,y-ak \quad\ldots\ldots\ldots\ldots (3),$$
$$x^2+y^2 = ax-ay \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

The radical axes of (1), (2), (3) with respect to (4) are

$$\begin{aligned}(a-h)\,x-(a+h)y \qquad\quad &= 0,\\ (-a+h+k)\,x+(a-h+k)\,y-hk &= 0,\\ kx+(2a-k)y \quad -ak &= 0,\end{aligned}$$

and the condition that these lines meet in a point reduces to $\dfrac{1}{a}+\dfrac{1}{h}=\dfrac{2}{k}$; therefore when h, k, a are in harmonical progression, the circles meet in a point.

2117. (Proposed by R. Tucker, M.A.)—A point P is given (1) on the base of a triangle, (2) on a circle; draw a transversal through another given point Q to cut (1) or (2) so that the intercept may subtend a given or a maximum angle at P.

Solution by James Dale.

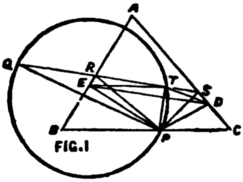


FIG. 1

1. Let QRS be the required transversal cutting the sides in R and S so that RPS is equal to the given angle. On PQ as diameter describe a circle, cutting the transversal in T; draw PD, PE perpendicular to AC, AB; and join PT, DE.

Then the angles SPT, RPT are respectively equal to ADT, AET,

therefore $\angle$ BAC + ADT + AET = BAC + RPS = DTE = a constant.

Hence the following construction. From P draw perpendiculars on the sides of the triangle; on the line joining the feet of these perpendiculars describe a segment of a circle containing an angle equal to the sum of the given angle and BAC; and join Q to either of the points in which the circle is cut by the circle described on PQ as diameter. The maximum angle is the supplement of BAC, the arc DTE in this case coinciding with DE.

2. In the case of the circle, draw any chord CD so that CPD may be equal to the given angle; describe a concentric circle touching CD, and from Q draw a tangent to this latter circle, and cutting the given circle in RS. Then since chords equidistant from the centre of a circle are equal, and equal chords subtend equal angles, it follows that the angle RPS is equal to the given angle.

The maximum angle is equal to the angle in the segment formed by joining QP.

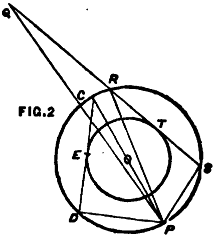


FIG. 2

2429. (Proposed by R. Tucker, M.A.)—An ellipse (A) and hyperbola (B) have the same transverse and conjugate diameters; prove that the polar of any point on (B) with reference to (A) touches (B).

I. *Solution by* James Dale.

Let a, b be the transverse and conjugate axes; then taking for axes of reference the asymptotes of the hyperbola, which are a pair of equal conjugate diameters of the ellipse, the equations of the two curves are

$$x^2+y^2=\tfrac{1}{2}(a^2+b^2),\quad xy=\tfrac{1}{4}(a^2+b^2)\ \ldots\ldots\ldots\ldots\ldots(\text{A, B}).$$

The polar of any point (h, k) with respect to (A) is $hx+ky=\frac{1}{2}(a^2+b^2)$, and this polar touches B if $hk=\frac{1}{4}(a^2+b^2)$, that is, if (h, k) is a point on B; therefore (B) is its own reciprocal with respect to (A).

II. *Solution by the* Proposer.

Let the ellipse and hyperbola be given by the equations

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ \frac{x^2}{a^2}-\frac{y^2}{b^2}=1\ \ldots.(\text{A, B}).$$

Now the polar of any point (x', y') on (B) with reference to (A) will have for its equation

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=1,$$

the envelope of which is readily found to be the hyperbola (B) or its conjugate.

If the conics be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \;\ldots\ldots (\text{B}),$$

$$a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c' = 0 \;\ldots\ldots (\text{A}),$$

the curve envelope will be given as the result of the elimination of (x', y') (a point on B) between (B) and

$$x'(a'x + h'y + g') + y'(h'x + b'y + f') + g'x + f'y + c' = 0.$$

The resulting equation is

$$\begin{aligned} &x'\left[(ab' - hh')y + (ah' - a'h)x + af' - bg'\right] \\ +&y'\left[(hb' - h'b)y + (hh' - a'b)x + hf' - bg'\right] \\ +&\quad (b'g - h'f)y + (h'g - a'f)x + f'g - fg' = 0. \end{aligned}$$

2438. (Proposed by R. Tucker, M.A.)—Two parabolas turned in opposite directions have the same axis and coincident vertices; if from the vertex as centre, and diameter a mean proportional between the parameters of the curves, a circle be described, prove that the polar of any point on one with reference to the circle touches the other.

I. *Solution by* W. H. Laverty; H. Tomlinson; *and others.*

Let the equations to the three curves be

$$y^2 = 4ax, \quad y^2 = -4bx, \quad x^2 + y^2 = 4ab \;\ldots\ldots\ldots\ldots (1, 2, 3).$$

Then the polar of (x', y') with reference to (3) is

$$yy' = -xx' + 4ab \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4);$$

and if (4) is coincident with $y = mx - \frac{b}{m}$ which is a tangent to (2), we have

$m = -\frac{x'}{y'}$, and $-\frac{b}{m} = \frac{4ab}{y'}$; or eliminating m, $y'^2 = 4ax'$, and (x', y') is a point on (1).

II. *Solution by the* Proposer.

Let $y^2 = 4ax$, and $x^2 + y^2 = r^2 \;\ldots\ldots\ldots\ldots\ldots\ldots (1, 2)$, be the equations to the parabola and circle; then the polar of any point (x', y') on (1) with reference to (2) will be given by

$$xx' + yy' = r^2 \;\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

the envelope of which is readily found to be $y^2 = -\frac{r^2}{a}x$, and the product of the parameters is $= 4a\left(\frac{r^2}{a}\right) = 4r^2$, which proves the property stated.

1985. (Proposed by Matthew Collins, B.A.)—Find what integral value of x renders $\frac{3x^7-29}{547}$ = a whole number, or else prove it impossible.

Solution by Samuel Bills.

If $\frac{3x^7-29}{547}$ = a whole number be possible for any very *large* numbers, it is well known that it will also be possible for some either positive or negative values of x *less* than $\frac{547}{2}$, that is for some value of x not *greater* than 273.

Now by using various artifices and contractions, I have tried *all* the numbers for x, both positive and negative, as far as 273, and none of them satisfy the required condition; I therefore conclude that the relation is impossible.

2339. (Proposed by T. Cotterill, M.A.)—1. If two triangles in a plane are similar, but right and left in position; then the lines drawn from the angular points of one triangle parallel, perpendicular, or isoclinal to the corresponding sides of the other triangle, meet in a point. Find its locus, if the first triangle remain fixed, while the second moves in any manner in the plane.

2. Analytically, we may assume the sums of the corresponding angles to vanish.

Solution by Samuel Roberts, M.A.

This proposition depends on the movement in rotation of the second triangle, and not on its translation in space. Moreover, if the lines corresponding to a given position of the second triangle are drawn (say) perpendicular to its sides, those lines will be parallel to sides of the second triangle in another position, and isoclinal in any position of that triangle. Let us take lines parallel to the sides of the second triangle. Denote the drawn lines by 1′, 2′, 3′, and the sides of the first triangle (fixed) by 1, 2, 3; then we have

$$\angle(1'2') = \angle(12), \quad \angle(2'3') = \angle(23), \quad \angle(3'1') = \pi - \angle(31),$$

supposing the figures are appropriated as the relative positions of the triangles require. It is very easy to see that the locus of the points is the circumscribing circle.

If we refer the triangles to two independent axes (x, y), we find that (A, B, C), (A′, B′, C′) take the forms $[\gamma-\beta, \pi-(\gamma-\alpha), (\beta-\alpha)]$, $[\beta-\gamma, \pi-(\alpha-\gamma), \alpha-\beta]$, and 2π produces the same results as 0.

2471. (Proposed by W. S. B. WOOLHOUSE, F.R.A.S.)—Let a given surface, having a convex boundary *of any form whatever*, be referred to its centre of gravity and the principal axes of rotation situated in its plane; and, corresponding to an abscissa x, let y, y' be the respective distances of the boundary above and below the axis; then, if h, k denote the radii of gyration round the axes, and M the total area, we shall have

$$h^2=\int x^2dx\,\frac{y+y'}{\text{M}},\quad k^2=\tfrac{1}{3}\int dx\,\frac{y^3+y'^3}{\text{M}}.$$

Let also $\qquad \text{A}=\int x^2dx\,\dfrac{y^3+y'^3}{\text{M}},\quad \text{B}=\int xdx\,\dfrac{y\int y^2dx+y'\int y'^2dx}{\text{M}},$

and $\qquad \text{C}=\tfrac{1}{3}\text{A}+3\text{B}$; then

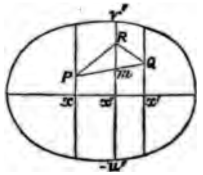
1. If *three* points be taken at random, on the given surface, the average area of the triangle connecting them, in parts of the total area, is $(\Delta)=\dfrac{h^2k^2+\text{C}}{\text{M}^2}$; and the average square of the area, expressed in parts of the square of the total area, is $(\Delta^2)=\dfrac{3}{2}\dfrac{h^2k}{\text{M}^2}$.

2. If *four* points be taken at random, the probability that the quadrilateral shall be reentrant $=4\,(\Delta)$.

3. If *five* points be taken at random on the surface, the probability of

a convex pentagon $=1-10\,(\Delta)+10\,(\Delta^2)$,
one reentrant point $=10\,(\Delta)-20\,(\Delta^2)$,
two reentrant points $=10\,(\Delta^2)$.

Solution by the PROPOSER.



In the direction of the axis of x the three points may be taken in the order P, R, Q, as in the diagram, since all other arrangements will eventually be included, if we afterwards suppose the position of the points to be permuted, and in consequence multiply the result by 6. Let the coordinates of P, Q, R be (x, y), (x', y'), (x'', y''), and let β denote the ordinate of the point m in which the side PQ is intersected by y'', produced if necessary; then we shall have

$$\beta=\frac{y\,(x'-x'')+y'(x''-x)}{x'-x},\quad \Delta=\frac{x'-x}{2}\,(y''-\beta),\ \int\Delta\,dy''=\frac{x'-x}{4}\,(y''-\beta)^2.$$

When R is above the point m, or $y''>\beta$, the integral gives $\frac{1}{4}\,(x'-x)\,(v''-\beta)^2$; and when R is below m, the integral, made also positive, is $\frac{1}{4}\,(x'-x)\,(u''+\beta)^2$, where v'', u'' refer to the upper and lower boundary. Therefore when R, or y'', varies from boundary to boundary, the value of the integral is

$$\int\Delta\,dy''=\tfrac{1}{4}\,(x'-x)\,\left\{(v''-\beta)^2+(u''+\beta)^2\right\}$$
$$=\tfrac{1}{4}\,(x'-x)\,\left\{v''^2+u''^2-2\,(v''-u'')\,\beta+2\beta^2\right\}.$$

To abbreviate, put $v''^2+u''^2=2\gamma''^2,\ v''-u''=2\epsilon'',\ v''+u''=2\mu''\ldots.(a)$;
Then $\int\Delta\,dy''=\frac{1}{2}(x'-x)\gamma''^2-(x'-x)\beta\epsilon''+\frac{1}{2}(x'-x)\beta^2=$

$$\tfrac{1}{2}(x'-x)\gamma''^2-\left\{y(x'-x'')+y'(x''-x)\right\}\epsilon''+\frac{\left\{y(x'-x'')+y'(x''-x)\right\}^2}{2(x'-x)};$$

$$\therefore \iint\Delta\,dx''dy''=\tfrac{1}{2}(x'-x)\int\gamma''^2dx''-(y'-y)\int\epsilon''x''\,dx''-(yx'-y'x)\int\epsilon''dx''$$
$$+\frac{\left\{y(x'-x'')+y'(x''-x)\right\}^3}{6(x'-x)(y'-y)}.$$

Now by hypothesis, as regards the abscissas, the point R lies between P and Q, so that the limits of x'' are x and x'. On taking the preceding integral between the limits, $x''=x\ldots x'$, the last term gives

$$\frac{y'^3(x'-x)^3-y^3(x'-x)^3}{6(x'-x)(y'-y)}=\frac{(x'-x)^2}{6}(y'^2+yy'+y^2).$$

Therefore, between the stated limits, we have

$$\iint\Delta\,dx''\,dy''=\frac{x'-x}{2}\left(\int\gamma'^2dx'-\int\gamma^2dx\right)-(y'-y)\left(\int\epsilon'x'dx'-\int\epsilon x\,dx\right)$$
$$-(yx'-y'x)\left(\int\epsilon'\,dx'-\int\epsilon\,dx\right)+\frac{(x'-x)^2}{6}(y'^2+yy'+y^2).$$

Again, the limits of y are $-u'\ldots.v'$;

$$\therefore\ \iiint\Delta\,dy'\,dx''\,dy''=\mu'(x'-x)\left(\int\gamma'^2dx'-\int\gamma^2dx\right)$$
$$-2\mu'(\epsilon'-y)\left(\int\epsilon'x'\,dx'-\int\epsilon x\,dx\right)$$
$$-2\mu'(yx'-\epsilon'x)\left(\int\epsilon'\,dx'-\int\epsilon\,dx\right)$$
$$+2\mu'\frac{(x'-x)^2}{6}\left(\frac{v'^2-v'u'+u'^2}{3}+\epsilon'y+y^2\right).$$

Also the limits of y are $-u\ldots.v$;

$$\therefore\ \iiiint\Delta\,dy\,dy'\,dx''\,dy''=4\mu\mu'\frac{x'-x}{2}\left(\int\gamma'^2dx'-\int\gamma^2dx\right)$$
$$-4\mu\mu'(\epsilon'-\epsilon)\left(\int\epsilon'x'\,dx'-\int\epsilon x\,dx\right)$$
$$-4\mu\mu'(\epsilon x'-\epsilon'x)\left(\int\epsilon'dx'-\int\epsilon\,dx\right)$$
$$+4\mu\mu'\frac{(x'-x)^2}{6}\left(\frac{v^2-vu+u^2}{3}+\epsilon\epsilon'+\frac{v'^2-v'u'+u'^2}{3}\right)$$

(or, putting $\frac{v^2-vu+u^2}{3}=\lambda^2,\ \frac{v'^2-v'u'+u'^2}{3}=\lambda'^2$, it becomes)

$$=2\mu\mu'\left(x'\int\gamma'^2\,dx'+x\int\gamma^2\,dx\right)-2\mu\mu'\left(x\int\gamma'^2\,dx'+x'\int\gamma^2\,dx\right)$$
$$-\ 4\mu\mu'\left(\epsilon'\int\epsilon'x'\,dx'+\epsilon\int\epsilon x\,dx\right)+4\mu\mu'\left(\epsilon\int\epsilon'x'\,dx'+\epsilon'\int\epsilon x\,dx\right)$$
$$-\ 4\mu\mu'\left(\epsilon x'\int\epsilon'\,dx'+\epsilon'x\int\epsilon\,dx\right)+4\mu\mu'\left(\epsilon'x\int\epsilon'\,dx'+\epsilon x'\int\epsilon\,dx\right)$$
$$+\frac{2\mu\mu'}{3}\left\{(x^2\lambda'^2+x'^2\lambda^2)+(x'^2\lambda'^2+x^2\lambda^2)+\epsilon\epsilon'(x'^2+x^2)-2xx'(\lambda'^2+\lambda^2)-2xx'\epsilon\epsilon'\right\}.$$

This expression is a symmetrical function of x, x', and it is here purposely arranged in pairs of terms of the form

$$X' = F'x\,f'x'+f'x\,F'x',$$

each of which pairs will, on further integration, give

$$\int X\,dx' = F'x\,fx' + f'x\,Fx' \quad (x' = x) \;= F'x\,fx + Fx\,f'x,$$

and therefore $$\iint X\,dx\,dx' = Fx\,.\,fx.$$

Hence, by observing this formula, the remaining double integration [illegible]f the foregoing expression is directly effected, and we get

$$\iiiint\!\!\iint \Delta\,dx\,dx'\,dy\,dy'\,dx'\,dy'' =$$

$$2\int\mu\,dx\,.\int\left(\mu x\,dx\int\gamma^2\,dx\right) - 2\int\mu x\,dx\,.\int\left(\mu\,dx\int\gamma^2\,dx\right)$$

$$-4\int\mu\,dx\,.\int\left(\mu\epsilon\,dx\int\epsilon x\,dx\right) + 4\int\mu\epsilon\,dx\,.\int\left(\mu\,dx\int\epsilon x\,dx\right)$$

$$-4\int\mu\epsilon\,dx\,.\int\left(\mu x\,dx\int\epsilon\,dx\right) + 4\int\mu x\,dx\,.\int\left(\mu\epsilon\,dx\int\epsilon\,dx\right)$$

$$+\tfrac{2}{3}\Big\{\int\mu x^2\,dx\,.\int\mu\lambda^2\,dx + \int\mu\,dx\,.\int\mu\lambda^2x^2\,dx + \int\mu\epsilon\,dx\,.\int\mu\epsilon\,x^2 - \quad dx$$

$$-2\int\mu x\,dx\,.\int\mu\lambda^2x\,dx - \int\mu\epsilon x\,dx\,.\int\mu\epsilon x\,dx\Big\}\ldots\ldots \quad (b).$$

According to the notation adopted in the question, M is the total area, [illegible]nd regarding the surface dynamically as a thin lamina, h, k are the radi[illegible]i of gyration round the axes;

$$\therefore \quad M = \iint dx\,dy \;=\; \int dx\,(v+u) \;=\; 2\int\mu\,dx,$$

$$M\,.\,h^2 = \iint x^2\,dx\,dy = \int x^2\,dx\,(v+u) = 2\int\mu x^2\,dx,$$

$$M\,.\,k^2 = \iint y^2\,dx\,dy = \tfrac{1}{3}\int dx\,(v^3+u^3) = 2\int\mu\lambda^2\,dx.$$

In all that precedes, the position of the coordinate axes is wholly unrestricted. If the origin be at the centre of gravity, we shall also have

$$0 = \iint x\,dx\,dy = \int x\,dx\,(v+u) = 2\int\mu x\,dx,$$

$$0 = \iint y\,dx\,dy = \tfrac{1}{2}\int dx\,(v^2-u^2) = 2\int\mu\epsilon\,dx.$$

Also, if the axes be principal axes of rotation, we shall have furthermore

$$0 = \iint xy\,dx\,dy = \tfrac{1}{2}\int x\,dx\,(v^2-u^2) = 2\int\mu\epsilon x\,dx.$$

The integral we have found, therefore, reduces down to the following:

$$\iiiint\!\!\iint(\quad) = M\int\left(\mu x\,dx\int\gamma^2dx\right) - 2M\int\left(\mu\epsilon\,dx\int\epsilon x\,dx\right) + \tfrac{1}{6}M^2h^2k^2 + \tfrac{1}{3}M\int\mu\lambda^2x^2dx.$$

But $$\int\left(\mu\epsilon\,dx\int\epsilon x\,dx\right) + \int\left(\epsilon x\,dx\int\mu\epsilon\,dx\right) = \int\mu\epsilon\,dx\,.\int\epsilon x\,dx = 0$$

gives $$\int\left(\mu\epsilon\,dx\int\epsilon x\,dx\right) = -\int\left(\epsilon x\,dx\int\mu\epsilon\,dx\right);$$

$$\therefore \iiiint\!\!\iint(\quad) = M\int\left(\mu x\,dx\int\gamma^2\,dx\right) + 2M\int\left(\epsilon x\,dx\int\mu\epsilon\,dx\right)$$

$$+\tfrac{1}{6}M^2h^2k^2 + \tfrac{1}{3}M\int\mu\lambda^2x^2\,dx$$

$$= \tfrac{1}{4}M\int\Big\{(v+u)\,x\,dx\int(v^2+u^2)\,dx\Big\} + \tfrac{1}{4}M\int\Big\{(v-u)\,x\,dx\int(v^2-u^2)\,dx\Big\}$$

$$+\tfrac{1}{6}M^2h^2k^2 + \tfrac{1}{18}M\int x^2dx\,(v^3+u^3)$$

$$= \tfrac{1}{2}M\int\left(vx\,dx\int v^2\,dx\right) + \tfrac{1}{2}M\int\left(ux\,dx\int u^2\,dx\right)$$

$$+\tfrac{1}{6}M^2h^2k^2 + \tfrac{1}{18}M\int x^2dx\,(v^3+u^3)\ldots(c).$$

By now permuting the three points, so as to obtain every position and combination, the sum of the areas is $6\iiint\!\!\iiint(\quad)$; and as each point traverses the area M, the number of positions is M^3. Therefore the average area is $\dfrac{6\iiint\!\!\iiint(\quad)}{M^3}$; and when expressed in parts of the total area, it is $(\Delta) = \dfrac{6\iiint\!\!\iiint(\quad)}{M^4}$. Hence, substituting y, y', respectively, for v, u, conformably with the notation in the question, we finally get

$$(\Delta) = \frac{3\int(y dx \int y^2 dx) + 3\int(y'x\, dx \int y'^2 dx)}{M^3} + \frac{h^2k^2}{M^2} + \frac{1}{3M^3}\int x^2\, dx\, (y^3 + y'^3)$$

$$= \frac{3B}{M^2} + \frac{h^2k^2}{M^2} + \frac{1}{3}\cdot\frac{A}{M^2} = \frac{h^2k^2 + C}{M^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (d),$$

which establishes the formulæ first stated.

To determine (Δ^2), the average value of Δ^2 divided by M^2, we have

$$\Delta = \tfrac{1}{2}(x'-x)(y''-\beta) = \tfrac{1}{2}y''(x'-x) - \tfrac{1}{2}y(x'-x'') - \tfrac{1}{2}y'(x''-x)$$
$$= \tfrac{1}{2}y''(x'-x) - \tfrac{1}{2}x''(y'-y) + \tfrac{1}{2}(xy'-x'y)$$
$$\therefore\quad \Delta^2 = \tfrac{1}{4}x''^2(y'-y)^2 + \tfrac{1}{4}y''^2(x'-x)^2 + \tfrac{1}{4}(xy'-x'y)^2$$
$$+ Hx'' + Ky'' + Lx''y'' \quad \ldots. (e),$$

where H, K, L are factors independent of the point (x'', y''), which factors will go out on integration with respect to that point. The average value of Δ^2 when one point $(x''y'')$ alone varies is

$$\Delta^2_{(1)} = \frac{\iint \Delta^2\, dx''\, dy''}{M} = \frac{\iint (e)\, dx''\, dy''}{M}.$$

Now, as h, k are the radii of gyration round the axes of x, y, we have

$$\iint x''^2\, dx''\, dy'' = M\,.\,h^2, \quad \iint y''^2\, dx''\, dy'' = M\,.\,k^2;$$

and, as the figure is referred to its centre of gravity and the principal axes of rotation through that centre as axes of coordinates, we shall also have

$$\iint x''\, dx''\, dy'' = 0, \quad \iint y''\, dx''\, dy'' = 0, \quad \iint x''y''\, dx''\, dy'' = 0.$$

Therefore from (e) we immediately find

$$\Delta^2_{(1)} = \tfrac{1}{4}h^2(y'^2 - 2yy' + y^2) + \tfrac{1}{4}k^2(x'^2 - 2xx' + x^2) + \tfrac{1}{4}(x^2y'^2 - 2xy\,x'y' + x'^2y^2)$$
$$\ldots\ldots (f).$$

Similarly the average value of Δ^2, when two points vary on the surface, is

$$\Delta^2_{(2)} = \frac{\iint (f)\, dx'\, dy'}{M} = \tfrac{1}{4}h^2(k^2_{*} + y^2) + \tfrac{1}{4}k^2(h^2_{*} + x^2) + \tfrac{1}{4}(k^2x^2_{*} + h^2y^2)$$
$$= \frac{h^2k^2}{2}\left(\frac{x^2}{h^2} + \frac{y^2}{k^2} + 1\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (g).$$

And, when all three points vary on the surface,

$$\Delta^2_{(3)} = \frac{\iint (g)\, dx\, dy}{M} = \frac{h^2k^2}{2}(1+1+1) = \frac{3}{2}h^2k^2 \quad \ldots\ldots\ldots\ldots (h).$$

Hence, dividing by M^2, we get $(\Delta^2) = \frac{3}{2} \cdot \frac{h^2k^2}{M^2}$ ....(k).

Now suppose *four* points P, Q, R, S to be taken on the surface. If the fourth point S be taken at random, since it may indiscriminately fall upon any point of the surface (M), the probability that it shall fall within the triangle PQR is evidently accurately expressed by the fraction $\frac{\text{triangle PQR}}{M}$. And when P, Q, R take all positions on the given surface, this probability obviously becomes $= \frac{\text{average triangle PQR}}{M} = (\Delta)$. Hence, as each point may be separately considered as the fourth, we have, generally,

$$\begin{aligned} (\Delta) &= \text{prob. of S falling in PQR,} \\ &= \quad ,, \quad \text{P} \quad ,, \quad \text{QRS,} \\ &= \quad ,, \quad \text{Q} \quad ,, \quad \text{RSP,} \\ &= \quad ,, \quad \text{R} \quad ,, \quad \text{SPQ.} \end{aligned}$$

These separate probabilities are identical in value, since the points admit of being permuted. The four conditions implied are also individually exclusive, and the sum of the values obviously makes up the complete probability that the quadrilateral shall be reentrant. Therefore if four points be taken at random on the given surface, the probability that the quadrilateral shall be reentrant $= 4(\Delta)$; that is, four times the average triangle expressed in parts of the total area.

Lastly, suppose that *five* points P, Q, R, S, T are taken at random on the given surface. Four assigned points P, Q, R, S will be the corners of a quadrilateral that is either convex or reentrant. If PQRS be convex, its area will be half the sum of the component triangles, viz.

$$\text{area} = \tfrac{1}{2}(\text{PQR} + \text{QRS} + \text{RSP} + \text{SPQ}) \text{ ....} (l),$$

and the fifth point T will be reentrant if it fall within this area. If PQRS be a reentrant quadrilateral, the area expressed by (l) will be that of the triangle which includes the reentrant point; and again the fifth point T will also be reentrant if it fall within this particular area. Hence the probability of the fifth point T being reentrant is in all cases equal to the area (l) divided by the total area; and when the points P, Q, R, S severally take all positions on the surface, this last-mentioned quotient becomes $= 2(\Delta)$. The other four points having each of them the same probability of being reentrant, we find by collecting these several probabilities that the total probability that one of the points shall be reentrant plus twice the probability that a pair of points shall be reentrant is equal to $10(\Delta)$.

Again, the probability that S, T shall be a pair of reentrant points is evidently $\frac{\text{PQR}}{M} \times \frac{\text{PQR}}{M} = \frac{(\text{PQR})^2}{M^2}$. When P, Q, R take all positions, this becomes $= (\Delta^2)$; and a like probability will obtain for every pair of the five points. Therefore the probability of a pair of reentrant points is $= 10(\Delta^2)$.

Hence also the probability of only one reentrant point $= 10(\Delta) - 20(\Delta^2)$. And the probability of a convex pentagon $= 1 - 10(\Delta) + 10(\Delta^2)$.

NOTE.—When the axis of x is diametral, then

$$h^2 = \frac{2}{M}\int yx^2dx, \qquad k^2 = \frac{2}{3M}\int y^3dx;$$

$$A = \frac{2}{M}\int y^3x^2dx, \qquad B = \frac{2}{M}\int yxdx\int y^2dx.$$

And in the formula for B, since $\int yxdx \,.\, c = 0$, the inner integral may be estimated from any epoch.

If the axis of y be also diametral, the calculation of the integrals may be further simplified by taking them for only one quadrant and doubling the results just stated.

Since the values depend upon the distribution of areas alone, it is evident that they will remain unchanged when the figure is modified in any way by orthogonal projection.

As examples of the practical application of the formulæ, a table is annexed showing the numerical results for four principal figures.

Examples.

Form of Boundary.	Triangle.	Square or Rectangle.	Regular Hexagon.	Circle or Ellipse.
Value of (Δ)	$\frac{1}{12}$	$\frac{11}{144}$	$\frac{289}{3888}$	$\frac{35}{48\pi^2}$
Value of (Δ^2)	$\frac{1}{72}$	$\frac{1}{96}$	$\frac{25}{2592}$	$\frac{3}{32\pi^2}$
Prob. of a reent. quadrilateral ..	$\frac{1}{3}$	$\frac{11}{36}$	$\frac{289}{972}$	$\frac{35}{12\pi^2}$
Prob. of a convex quadrilateral ..	$\frac{2}{3}$	$\frac{25}{36}$	$\frac{683}{972}$	$1-\frac{35}{12\pi^2}$
Prob. of a convex pentagon	$\frac{11}{36}$	$\frac{49}{144}$	$\frac{1373}{3888}$	$1-\frac{305}{48\pi^2}$
Prob. of one reentrant point....	$\frac{20}{36}$	$\frac{80}{144}$	$\frac{2140}{3888}$	$\frac{260}{48\pi^2}$
Prob. of two reentrant points ..	$\frac{5}{36}$	$\frac{15}{144}$	$\frac{375}{3888}$	$\frac{45}{48\pi^2}$
Value of (Δ)	·08333	·07639	·07433	·07388
Value of (Δ^2)	·01389	·01042	·00964	·00950
Prob. of a reent. quadrilateral ..	3333	·3056	·2973	·2955
Prob. of a convex quadrilateral..	6667	·6944	·7027	·7045
Prob. of a convex pentagon	·3056	·3403	·3531	·3562
Prob. of one reentrant point. ..	·5555	·5555	·5504	·5488
Prob. of two reentrant points ..	·1389	·1042	·0965	·0950

2472. (Proposed by Professor Cayley.)—Through four points on a circle to draw a conic, such that an axis may pass through the centre of the circle.

Solution by the PROPOSER.

Let the equation of the conic be $(a, b, c, f, g, h)(x, y, 1)^2 = 0$, then if as usual the inverse coefficients are represented by (A, B, C, F, G, H), the equation of the two axes is

$$(a-b)(Cx-G)(Cy-F)+h[(Cx-G)^2-(Cy-F)^2] = 0,$$

whence if an axis pass through the origin

$$(a-b)\,FG+h(G^2-F^2) = 0.$$

Consider now the circle $x^2+y^2-1 = 0$ and on it the four points in which it is intersected by the conic $(a, b, c, f, g, h)(x, y, 1)^2 = 0$; then for any conics through the four points we have

$$(a, b, c, f, g, h)(x, y, 1)^2+\lambda(x^2+y^2-1) = 0;$$

so that, taking this for the equation of the required conic, and representing it by

$$(a', b', c', f', g', h')(x, y, 1)^2 = 0,$$

the values of the coefficients are

$$a' = a+\lambda,\ b' = b+\lambda,\ c'=c-\lambda,\ f'=f,\ g'=g,\ h' = h,$$

and we thence have

$$F' = F-\lambda f,\ G' = G-\lambda g,\ a'-b' = a-b,\ h' = h.$$

The required relation is $(a'-b')\,F'G'+h'(G'^2-F'^2) = 0$,

that is $(a-b)(F-\lambda f)(G-\lambda g)+h\left\{(G-\lambda g)^2-(F-\lambda f)^2\right\} = 0$,

a quadric equation in λ; and substituting for λ each of its two values, we have the two required conics

$$(a, b, c, f, g, h)(x, y, 1)^2+\lambda(x^2+y^2-1) = 0,$$

for each of which an axis passes through the centre of the circle.

2473. (Proposed by Professor SYLVESTER.)—Show (1) how to plant 81 trees so as to form 800 rows of 3 trees in a row; (2) how to plant 10 trees so as to form 10 rows of three trees in a row.

Solution by the PROPOSER.

1. Take any cubic curve and P any point on it. Draw PQ a tangent at P cutting the curve in Q, QR a tangent at Q cutting the curve in R, join PR cutting the curve in S, QS cutting it in T, PT cutting it in U, QU cutting it in V, and so on, until m points are obtained at which trees are supposed to be planted. Then according as m is of the form $4k+3$, $4k+2$, $4k+1$, $4k$, the number (μ) of collineations between the points taken 3 and 3 together may, by the Theory of Unicursal Derivation (laid before the London Mathematical Society by the Proposer), be shown to be $2k^2+2k$, $2k^2+k$, $2k^2$, $2k^2-k$ respectively. In the first case proposed, we have $k = 20$, and $m = 4k+1$; therefore $\mu = 2k^2 = 800$.

2. Another interesting case pointed out by Professor CAYLEY is where $k = 2$, $m = 4k+2 = 10$, and $\mu = 2k^2+k = 10$. This is the case of 10 trees

planted so as to form 10 rows, a variation upon the old problem of planting 9 trees so as to form 9 rows.

NOTE.—If we give to P, Q, R, S, T, &c., the names 1, 2, 4, 5, 7, &c., leaving out multiples of 3, then any 3 points whose numerical names λ, μ, ν satisfy the equation $\lambda^4+\mu^4+\nu^4-2\lambda^2\mu^2-2\mu^2\nu^2-2\nu^2\lambda^2=0$ will lie in the same straight line.

[In the *Mathematician*, Vol. II., p. 278, Mr. WOOLHOUSE has shown that 15 points may be placed so as to form 26 rows of 3 in a row.]

1927. (Proposed by W. S. BURNSIDE, M.A.)—To find the conic of least eccentricity that can be drawn through four given points.

I. *Solution by* H. R. GREER, B.A.

Let the points be defined by the intersections of two conics U and V which are given in the form $\frac{l}{x}+\frac{m}{y}+\frac{n}{z}=0$, and $\frac{l'}{x}+\frac{m'}{y}+\frac{n'}{z}=0$; and let the coordinates of their fourth point of intersection be proportional to $(\alpha^{-1}, \beta^{-1}, \gamma^{-1})$; so that we shall have $l\alpha+m\beta+n\gamma=0$, $l'\alpha+m'\beta+n'\gamma=0$. Now it is demonstrable that there exists a method of geometrical derivation according to which a line being represented by a conic through three fixed points, the line at infinity shall be represented by the circle through the same; and furthermore, that if the perpendiculars from any point on the sides of the fixed triangle be (α, β, γ), those from the corresponding point will be proportional to $(\alpha^{-1}, \beta^{-1}, \gamma^{-1})$; and also, that the angle between the asymptotes of any conic through the fundamental points will be equal to that in the segment cut off from the circumscribing circle by the line corresponding to the conic. Assuming these principles, which are built upon a particular method of *Quadric Inversion*, it is evident that, in the case before us, we must proceed as follows. Form the equation of the shortest chord of the circumscribing circle through (α, β, γ), and in the result write for (x, y, z) their reciprocals. Now the equation of the line joining (α, β, γ) to the centre of the circumscribing circle is $\begin{vmatrix} x & \alpha & \cos A \\ y & \beta & \cos B \\ z & \gamma & \cos C \end{vmatrix}=0$, say $Lx+My+Nz=0$; and that of the line through (α, β, γ) perpendicular to this is, say, $\begin{vmatrix} x & \alpha & \lambda \\ y & \beta & \mu \\ z & \gamma & \nu \end{vmatrix}=0$, where $\lambda=L-M\cos C-N\cos B$, and cyclically for μ and ν. But $L=\beta\cos C-\gamma\cos B$, and cyclically for M and N; therefore, after a little reduction,

$$\lambda=\alpha(\sin^2 B-\sin^2 C)+\beta\,.\sin A\sin B-\gamma\sin A\sin C;$$

therefore, writing $\sin A : \sin B : \sin C = a : b : c$, the equation of the required shortest chord is

$$\begin{vmatrix} x & \alpha & \alpha(b^2-c^2)+ab\beta-ca\gamma \\ y & \beta & \beta(c^2-a^2)+bc\gamma-ab\alpha \\ z & \gamma & \gamma(a^2-b^2)+ca\alpha-bc\beta \end{vmatrix} = 0.$$

The equation of the sought conic is got, consequently, by writing in the above for x, y, z, yz, zx, xy respectively; and for α, β, γ, $mn'-m'n$, &c. &c.

II. *Solution by the* PROPOSER.

If the four points do not form a reentrant quadrilateral, an ellipse can be described through the points. This being so, let

$U \equiv (a, b, c, f, g, h)(x, y, z)^2 = 0$, and $V = (a_1, b_1, c_1, f_1, g_1, h_1)(x, y, z)^2 = 0$

be the equations of any two conics passing through these points, then the equation of any other conic of the system is of the form $U + kV = 0$.

Now, put $A = bc-f^2$, &c.,

$$\theta \equiv A\sin^2 A + B\sin^2 B + C\sin^2 C + 2F\sin B\sin C + 2G\sin C\sin A + 2H\sin A\sin B,$$

$$\theta' \equiv a+b+c-2f\cos A-2g\cos B-2h\cos C;$$

also let θ_1 and θ'_1 be the similar quantities relative to V, and Θ, Θ' relative to $U+kV$. Then we have

$$\Theta \equiv \theta + k\phi + k^2\theta_1, \quad \Theta' \equiv \theta' + k\theta'_1,$$

where $\phi \equiv (bc_1+b_1c-2ff_1)\sin^2 A + (ca_1+c_1a-2gg_1)\sin^2 B$
$+ (ab_1+a_1b-2hh_1)\sin^2 C + 2(gh_1+g_1h-af_1-a_1f)\sin B\sin C$
$+ 2(hf_1+h_1f-bg_1-b_1g)\sin C\sin A + 2(fg_1+f_1g-ch_1-c_1h)\sin A\sin B.$*

Now, from the identity $(a, b, c, f, g, h)(x, y, z)^2 \equiv \lambda(X^2+Y^2-e^2Z)$, where $Z = 0$ is the equation of the directrix, and e the eccentricity, we have

$\theta' = \lambda(2-e^2)$, $\theta = \lambda^2(1-e^2)$, giving $\dfrac{\theta'^2-4\theta}{\theta} = \dfrac{e^4}{1-e^2}$; whence we determine k, so that the eccentricity of the conic $U+kV=0$ shall be a minimum by equating to zero the differential of $\dfrac{\Theta'^2}{\Theta}$ with respect to k. This being done, we find

$$(\theta'+k\theta'_1)(\phi+2k\theta_1) = 2\theta'_1(\theta+k\phi+k^2\theta_1),$$

or $\theta'\phi-2\theta\theta'_1 = k(\theta'_1\phi-2\theta'\theta_1)$, which determines k when the eccentricity e of the conic $U+kV=0$ is a minimum.

NOTE.—The condition that the four points ($U=0$, $V=0$) lie on a circle may be readily expressed in terms of θ, θ', θ_1, θ'_1, ϕ, as follows:—

In general if $U=0$ represents a circle $\theta'^2=4\theta$; therefore when $U+kV=0$ is the equation of a circle, $k^2(\theta_1'^2-4\theta_1)+2k(\theta'\theta'_1-2\phi)+\theta'^2-4\theta = 0$ determines k, and it is plain that k can have but one value, whence the condition required is

$$(\theta'^2-4\theta)(\theta_1'^2-4\theta_1) = (\theta'\theta'_1-2\phi)^2.$$

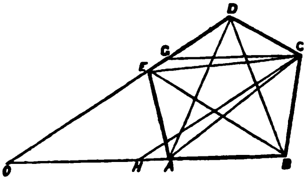
[* The geometrical meaning of ϕ is that the conics $U=0$, $V=0$ cut the line at infinity harmonically.]

2418. (Proposed by Dr. Shaw.)—If the vertices of a pentagon be A, B, C, D, E; and if a, b, c, d, e be the areas of the triangles EAB, ABC, BCD, CDE, DEA; show that the area P of the pentagon is given by the quadratic equation

$$P^2-(a+b+c+d)\,P+(ab+bc+cd+de+ea)=0.$$

I. *Solution by* Stephen Watson.

Produce any two of the sides, as BA, DE, to meet in O, and draw CG parallel to AB and CH to DE. Put $OA=a_1$, $OB=a_2$, $OE=b_1$, $OD=b_2$, $CG=\alpha$, $CH=\beta$, and $\sin\angle O=m$. Then



$$\begin{aligned} P &= \tfrac{1}{2}m\,(a_2\beta+b_2\alpha-a_1b_1)\\ &= \tfrac{1}{2}m\,(a_2b_2-a_1b_1)+c\ldots\ldots(1).\end{aligned}$$

$$\text{Also } \left.\begin{aligned} a &= \tfrac{1}{2}mb_1\,(a_2-a_1)\\ b &= \tfrac{1}{2}m\beta\,(a_2-a_1)\\ d &= \tfrac{1}{2}m\alpha\,(b_2-b_1)\\ e &= \tfrac{1}{2}ma_1\,(b_2-b_1)\end{aligned}\right\}\ldots\ldots(2).$$

From (2) we easily get

$$a_1=\frac{ea}{d},\ b_1=\frac{a\beta}{b},\ a_2=\frac{ea}{d}+\frac{2b}{m\beta},\ b_2=\frac{a\beta}{b}+\frac{2d}{m\alpha};$$

hence by substitution (1) becomes

$$P=b+d+\tfrac{1}{2}\left(\frac{ad+be-ae}{bd}\right)\alpha\beta m=a+c+e+\frac{2bd}{\alpha\beta m};$$

whence eliminating $\alpha\beta m$, we have, as required,

$$P^2-(a+b+c+d+e)\,P+(ab+bc+cd+de+ea)=0.$$

II. *Solution by* James Dale.

The six angles made at A by the lines AB, AC, AD, AE have their sines connected by the following well-known relation,

$$\sin BAC\,.\,\sin DAE+\sin BAE\,.\,\sin CAD=\sin BAD\,.\,\sin CAE;$$

whence, multiplying each term by ($\tfrac{1}{4}$AB . AC . AD . AE), we get

$$\triangle ABC\,.\,\triangle ADE+\triangle ABE\,.\,\triangle ACD=\triangle ABD\,.\,\triangle ACE;$$

or putting a_1, b_1, c_1, d_1, e_1 for the areas of the triangles CAD, DBE, ECA, ADB, BEC, we have the following conditions:—

$$\begin{array}{l|l} aa_1+be=c_1d_1\ldots\ldots(1) & a_1+b+e=P\ \ldots\ldots(6)\\ bb_1+ca=d_1e_1\ldots\ldots(2) & b_1+c+a=P\ \ldots\ldots(7)\\ cc_1+db=e_1a_1\ldots\ldots(3) & c_1+d+b=P\ \ldots\ldots(8)\\ dd_1+ec=a_1b_1\ldots\ldots(4) & d_1+e+c=P\ \ldots\ldots(9)\\ ee_1+ad=b_1c_1.\ \ldots.(5) & e_1+a+d=P\ldots\ldots(10).\end{array}$$

From (8) and (9), $c_1d_1=P^2-(b+c+d+e)\,P+(bc+cd+de+be)$

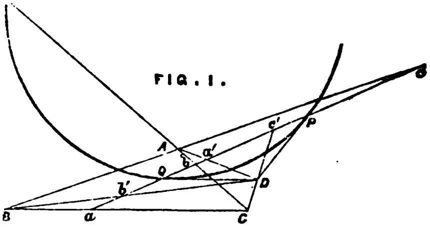
$$\begin{aligned} &=aa_1+be, \text{ by (1)}\\ &=aP-ab-ea+be, \text{ by (6)};\end{aligned}$$

therefore $P^2-(a+b+c+d+e)\,P+(ab+bc+cd+de+ea)=0.$

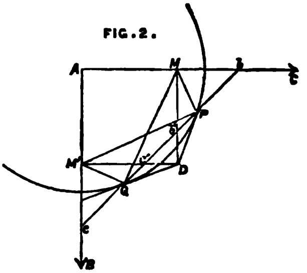
[For another solution see Hymer's *Trigonometry*, 3rd ed., p. 146.]

2376. (Proposed by T. COTTERILL, M.A.)—1. If ABC is a self-conjugate triangle of a conic, to which DP, DQ are tangents at P and Q, then any conic through ABCD cuts PQ harmonically. 2. Hence, if DM, DM′ are the perpendiculars on the axes, DM, DM bisect the angles PMQ, PM′Q.

Solution by JAMES DALE.



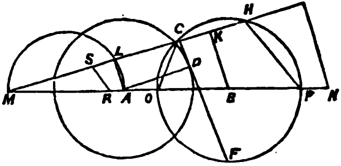
1. Let the chord PQ cut BC, CA, AB in a, b, c; join DA, DB, DC cutting PQ in a', b', c'; then the points aa', bb', cc' form a system in involution; and as each of the triangles Daa', Dbb', Dcc' is self-conjugate with respect to the conic, it follows that P, Q are the double points of the system aa' bb' cc'. If now any conic through ABCD cut PQ in P′, Q′, then these points also belong to the system aa' bb' cc', and therefore form a harmonic range with the double points P, Q.



2. Suppose now that AB, AC are the axes, then BC passes to infinity. Let PQ cut the axes in b, c, and DM, DM′ perpendicular respectively to AC, AB cut PQ in b', c'; join PM, QM; PM′, QM′; then PQ is divided harmonically in b, b'; and as the conjugate rays Mb, M′b' are at right angles, it follows that MP, MQ make equal angles with MD; so also M′C, M′C′ make equal angles with M′D.

1973. (Proposed by the Rev. R. TOWNSEND, M.A.)—Draw a chord of one of two given circles so as to touch and be cut in a given ratio at its point of contact with the other.

Solution by WILLIAM HOPPS.



Let A, B be the centres of the given circles, and suppose the tangent to the circle (A) at D, to meet the circle (B) at C and F, so that CD : DF $= m : n$, the given ratio. Draw AB, AD, and through C draw a line parallel to AD to meet BA produced at M, and the circle (B) at H; also draw AL, BK each perpendicular to CF. Then CD : DF $= m : n$, by hypothesis,

therefore CD : (CD + DF) = AL : CF $= m : (m+n)$,

therefore AL : $\frac{1}{2}$ CF (or BK) = MA : MB $= m : \frac{1}{2}(m+n)$;

hence MA : MB − MA (or AB) $= m : \frac{1}{2}(n-m) =$ a given ratio.

But AB is given, therefore MA is also given, and likewise a semicircle thereon passes through L, because MLA is a right angle; moreover LC = AD, which is given. We have, therefore, to draw a line from M to cut the semicircle on MA in L, and the circle (B) in C, so that the part LC intercepted by their circumferences shall be equal to AD, the radius of the given circle (A).

Produce MH until HI = LC or AD; at I draw a line perpendicular to MI to meet AB produced at N, and let O, P be the intersections of the circle (B) with MN. Then HI = LC and KH = CK; hence KI = LK, and therefore BN = AB; consequently N is a given point. Draw CO, HP, and divide MA at R, so that MN : MP = MA : MR; also draw RS parallel to HP, as in the diagram. Then we have

$$MI : MN = ML : MA,\ MP : MH = MR : MS,\ MN : MP = MA : MR \text{ (Con.)};$$

hence compounding these equal ratios we get

$$MI : MH = ML : MS, \text{ therefore } HI \text{ (or } LC) : MH = SL : MS;$$

therefore $$LC : SL = MH : MS = MP : MR.$$

Now LC, MR, MP are given, therefore SL is also given, consequently SC or MC − MS is given. But ∠ MRS = MPH = MCO, hence S, R, O, C are in the circumference of a circle, therefore MC . MS = MO . MR = a given space. That is, the difference and rectangle of MC, MS are given to determine them, which is a well-known elementary problem.

It may be observed that to draw from M a line to cut the semicircle on MA in L, and meet the circle B at C, so that LC shall be of given length, is a problem in Apollonius's lost treatise *De Inclinationibus*, restored by Ghetaldus and others. (See Bonnycastle's *Geometry*, 6th edition, p. 367; also Hutton's or Barlow's *Mathematical Dictionary*, under the name Apollonius.)

NOTE ON A CRITERION OF DIVISIBILITY. By M. JENKINS, B.A.

The rule for finding a criterion of divisibility by continued subtraction of digits, beginning at the left hand (given on p. 69 of this volume of the *Reprint*), may give rise to inconveniently high numbers to be considered as digits of the scale of notation: *e.g.* if the digits situated at the requisite interval be 9, 1, 9, 1, 9, 1, &c. The continued subtraction then gives successively −8, +17, −16, +25, &c.

The rule, however, may be put in a different form, and generalized thus:—

Let r be the base of the scale of notation; p the proposed divisor, supposed prime to r; N the dividend, a criterion of whose divisibility by p is required; and let $r^T - a$ be divisible by p, or $r^T \equiv a \pmod{p}$, where a may be supposed positive or negative, but of absolute magnitude not greater than $\frac{1}{2}p$; and T may be taken to be the least value of the index consistent with the congruence.

Mark off from the right hand of N divisions of T digits, and let $s_n, s_{n-1}, \dots s_1, s_0$ be the magnitudes of the divisions standing by themselves (s_n, the extreme left-hand division, consisting possibly of less than T digits): then $s_n a^n + s_{n-1} a^{n-1} + \dots + s_1 a + s_0$ is a criterion of the divisibility of N by p.

This criterion might be expressed to be the magnitude of the divisions considered as digits in the scale of a.

The correctness of the rule is evident from the fact that if s'_m be the local magnitude of the $(m+1)$th division,

$$s'_m = s_m r^{m\mathrm{T}};\ \text{and therefore } s'_m \equiv s_m a^m\ [\text{mod. } (r^{\mathrm{T}} - a)];$$

therefore $\mathrm{N} \equiv s_n a^n + \ldots + s_1 a_1 + s_0\ [\text{mod. } (r^{\mathrm{T}} - a)]$; *a fortiori* (mod. p).

Let t be the number of digits in the period of p^{-1}, expressed as an r-mal: then if $\mathrm{T} = t$, $a = +1$; and we have the rule, "Add all the divisions together for a criterion."

If t be even, and $\mathrm{T} = \frac{1}{2}t$, $a = -1$; then the rule becomes:—Subtract the sum of the divisions of even rank from the sum of those of uneven rank; the difference with the proper sign is congruential to the given number N.

The analogy of these two rules to the ordinary ones for 9 and 11 is obvious: by the continued application of them a number of not greater than t or $\frac{1}{2}t$ digits, as the case may be, can always, if t be known, be speedily found congruential to N. But the utility of the further application of the general rule to a number of t' digits, where t' is not $> t$ (t odd) or $\frac{1}{2}t$ (t even) is the probability of finding a congruence $r^{\mathrm{T}} \equiv a$ (mod. p) such that a and the greatest integer in $t' \div \mathrm{T}$ shall both be small.

Examples.—Find a criterion of divisibility by 7 and 13, and one for divisibility by 37 of

$$\begin{array}{r}
57,\ 002 \mid 637,\ 095 \mid 430,\ 019 \\
637,\ 095 \\
57,\ 002 \\
\hline
1124\ \ 116 \\
116\quad\ \ \\
\hline
1{,}008\quad\ \ \\
1\quad\ \ \\
\hline
7\quad\ \
\end{array}$$

$10^3 \equiv -1$ (mod. 7 or 13)

The given number is therefore divisible by 7, and leaves a least residue -7, or a remainder 6, after division by 13. For the divisor 37, we have, since $10^3 \equiv +1$ (mod. 37),

$$\mathrm{N} \equiv 1124 + 116 \equiv 124 \times 10 \equiv 13 \times 10 \equiv 19 \text{ (mod. 37)}.$$

Find a criterion of divisibility by 17 of

$$\begin{array}{rr}
53 \mid 7826 \mid 5918 \mid 4379 \mid 6210 & \mid 4389 \\
+53 & 7826 \\
\hline
6264 & 12215 \\
-5918 & 4379 \\
\hline
\cancel{3},45 & \mid \cancel{7}\cancel{8},36 \\
-6 & -156 \\
\hline
39 & -120 \\
 & 156 \\
\hline
 & 36
\end{array}$$

The original number is first reduced to 8 digits, since $10^8 \equiv -1$ (mod. 17); then, since $10^2 \equiv -2$, and $10^4 \equiv +4$ (mod. 17), we multiply 3 by -2 and add to 45; 78 by -2 and add to 36; 39 by 4 and add to -120; this process gives the criterion 36.

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1496. Prove that a triangular pyramid whose vertices are ABCD, and a parallelepiped formed from it as follows, have the same centre of gravity; viz. through any point in each of the opposite edges AB and CD draw straight lines parallel to the other edge, we thus get two parallel plane faces of the parallelepiped; two other parallel faces of it are similarly obtained from the opposite edges AC and DB; and the third pair of faces are obtained from the remaining two opposite edges AD and BC 71

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2244. Form eleven symbols into sets, five symbols in each set, so that every combination of four symbols shall appear once in the sets 35

2288. In Quest. 1705 (*Reprint*, Vol. III., p. 109), it is stated, that "if from the intersection of the diagonals of a quadrilateral

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inscribed in a circle perpendiculars be drawn on the sides, the quadrilateral formed by joining the feet of these perpendiculars is, of all quadrilaterals inscribed in the given one, the one of least perimeter." Discuss the more general case in which the given quadrilateral is not inscribable in a circle.................. 47

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2323. Determine the order, in the coefficients, of the conditions that three cubic equations may have a common root, the same being one of a pair of equal roots as to two of the given equations... 94

2344. Let P be any point on a rectangular hyperbola whose centre is O; then if with centre P and radius PO a circle be described, an infinite number of triangles can be inscribed in the circle, and circumscribed to the hyperbola. Let ABC be any one of these triangles; l, m, n the points where the sides BC, CA, AB touch the hyperbola: it is required to prove that the straight lines Al, Bm, Cn intersect on the circle 40

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2358. La condition qu'une conique divise harmoniquement les trois diagonales d'un quadrilatère circonscrit à une autre conique, coincide avec la condition que la première conique soit circonscrite à un triangle conjugué à l'autre 74

2362. Find the curves whose circles of curvature intercept upon a given line portions which (1) vary as the radii of curvature, (2) are constant 66

2367. If $x^n + b_1x^{n-1} + b_2x^{n-2} + \dots\dots + b_n = 0$ be an algebraical equation whose roots are $x_1, x_2, \dots\dots x_n$ show that

$$\begin{vmatrix} \frac{db_1}{dx_1}, & \frac{db_2}{dx_1}, & \frac{db_3}{dx_1} & \cdots & \frac{db_n}{dx_1} \\ \frac{db_1}{dx_2}, & \frac{db_2}{dx_2}, & \frac{db_3}{dx_2} & \cdots & \frac{db_n}{dx_2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{db_1}{dx_n} & \dots\dots & \dots\dots & \dots\dots & \frac{db_n}{dx_n} \end{vmatrix} = \sqrt{(\Delta)},$$

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2404. Let A′, B′, C′ be the centre of the escribed circles of a triangle ABC; I that of the inscribed circle; α, β, γ the middle points of the sides of the triangle A′B′C′. On the segments IA′, IB′, IC′ take X, Y, Z such that $\text{IX} = \frac{1}{4}\text{IA}'$, $\text{IY} = \frac{1}{4}\text{IB}'$, $\text{IZ} = \frac{1}{4}\text{IC}'$, then the circle through X, Y, Z will bisect each of the six segments IA, IB, IC, Iα, Iβ, Iγ.................................... 41

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2433. Five points being taken at random on the surface of a regular polygon of n sides as the apices of a pentagon, show that the probability of two of them being re-entrant is

$$\frac{5}{12}\left\{\frac{\cos\frac{2}{n}+2}{n\sin\frac{2\pi}{n}}\right\}^2.$$ 94

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2434. 1. Three different persons have each to name an integer not greater than n. The chance that the integers named are proportional to the sides of some real triangle is

$$\tfrac{1}{2}\left(1+\frac{1}{n^2}\right).$$

2. A person names a group of three integers (not necessarily different, but each one not greater than n). The chance that the integers named are proportional to the sides of some real triangle is

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2451. If A, B, C, D are the intersections of a conic by a circle, then the anti-points of A, B, and the anti-points of C, D, lie on a confocal conic.

N.B.— If AB, A′B′ intersect at right angles in a point O in such wise that $OA' = OB' = i.OA = i.OB$ [where $i = \sqrt{(-1)}$ as usual], then A′, B′ are the anti-points of A, B, and conversely 87

2464. From a point in the plane of a conic circumscribing a triangle let lines be drawn to meet its three sides parallel respectively to the diameters bisecting those sides. If the area of the triangle formed by lines joining the feet of these parallels be constant, the locus of the point from which they are drawn will be a similar and coaxial conic.................................. 82

2469. Transform the equation $x^4+px^3+qx^2+rx+s=0$, with roots $(\alpha, \beta, \gamma, \delta)$, to one whose roots are

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2470. Prove that

$$\frac{n-1}{1}\cdot\frac{2^2}{3}-\frac{(n-1)(n-2)}{1.2}\cdot\frac{2^3}{4}+\&c. = \frac{n+2}{n+1},\ (n \text{ even})$$ 26

2475. Trace the curve whose equation in triangular coordinates (x, y, z) is

$$\left(\frac{1-x}{x}\right)^{x(1-x)}\left(\frac{1-y}{y}\right)^{y(1-y)}\left(\frac{1-z}{z}\right)^{z(1-z)} = \epsilon^{p};$$

that is, the curve which is the locus of P in Question 2359 ... 17

2477. If there be described any curve APB upon the chord AB, the average area of all the triangles, formed by joining a point Q taken at random within the area APB and two fixed points C,

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D in the line AB, is equal to the area of the triangle formed by joining C, D and the centre of gravity of the area APB ... 112

2478. Deduce the first integrals

$$f+x\left(\frac{\mu}{r}-\frac{dy^2+dx^2}{dt^2}\right)+\frac{ydy+zdz}{dt}\cdot\frac{dx}{dt}=0,\ f'\ \text{\&c. \&c.}$$

(*Méc. Cél.* t. II., c. iii., § 18) directly from the three equations of motion

$$(1)\ldots\frac{d^2x}{dt^2}+\frac{\mu x}{r^3}=0,\ (2)\ldots\frac{d^2y}{dt^2}+\frac{\mu y}{r^3}=0,\ (3)\ldots\frac{d^2z}{dt^2}+\frac{\mu z}{r^3}=0;$$

these first integrals being merely shown by LAPLACE to satisfy the last three differential equations 24

2479. Determine m so that

$$\frac{-3b+am}{2a}+\frac{\sqrt{(9b^2-12ac-6abm-3a^2m^2)}}{2a}$$

shall be a root of the cubic $ax^3+3bx^2+3cx+d=0$ 47

2480. A circle is drawn round a triangle ABC; and from any point D, lines DB, DC are drawn cutting the circle in P and Q, whose feet-perpendicular lines with reference to ABC intersect in S; prove that the angle S is equal to the difference between the angles A and D 34

2481. Find the developments of $\frac{(\sin^{-1}x)^{2n-1}}{\Gamma(2n)}$ and $\frac{(\sin^{-2}x)^{2n}}{\Gamma(2n+1)}$, and show that they possess a property analogous to that noticed by Professor Sylvester (see Question 2447) as belonging to the developments of $(\sin^{-1}x)$ and $\frac{1}{2}(\sin^{-1}x)^2$ 56

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2484. If A, B, C denote the angles of a plane triangle, L, M, N known quantities, and $f(x)\equiv x+x^{-1}$, find the value of k obtained by eliminating x and y from the equations

$f(x)+2\cos A=kL,\ f(y)+2\cos B=kM,\ f(xy)+2\cos C=kN$... 33

2489. Through a point O, within a triangle ABC, parallels DE, FG, HI are drawn to BC, CA, AB, respectively; find the locus of O when the area of the hexagon DIFEHG is constant 25

2491. Solve, by definite integration, the equation

$$(x^3-1)\frac{d^3y}{dx^3}-3ax^2\frac{d^2y}{dx^2}+3a(a+1)x\frac{dy}{dx}-a(a+1)(a+2)y=0\ldots$$ 21

2493. 1. Given the conic $U=0$ (but observe that the *function* U contains implicitly an arbitrary constant factor which is *not* given) and also the conic $U+1=0$, to construct the conic $U+l=0$, where l is a given constant.

2. Given the conics $U=0$, $U+1=0$, $V=0$, $V+1=0$, and the constants θ, k, to construct the conic $\theta U+\theta^{-1}V+2k=0$... 82

2498. Let S_n represent the sum of the nth powers of the roots of of the cubic $ax^3+3bx^2+3cx+d=0$, and S'_n the result of

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interchanging a and d, b and c, in S_n; then prove that
$$a^{2n}S_{2n} = (a^nS_n)^2 + 2(-1)^{n+1}a^nd^nS'_n,$$
and hence show that
$$a^{12}S_{12} = (3a^4d^2 - 108a^3bcd - 54a^3c^3 + 729a^2b^2c^2 + 162a^2b^3d - 1458ab^4c + 729b^6)^2 - 2a^6(3d^4a^2 - 108d^3cba - 54d^3b^3 + 729b^2c^2d^2 + 162d^2c^3a - 1458dc^4b + 729c^6)$$ 62

2500. 1. If two points be taken at random within a circle, the chance that the line joining them will meet a given diameter at some point within the circle is $\frac{3\pi^2+16}{6\pi^2}$.

2. If three points be taken at random within a circle, the chance that these three points and the centre of the circle will form the angles of a re-entrant quadrilateral is $\frac{3\pi^2+16}{12\pi^2}$ 95

2502. If A, B, C, D be functions of x, y, z; and A_1, B_1, C_1, D_1 be certain constants; required the envelope of $A + \lambda B + \mu C + \nu D = 0$, when the variables λ, μ, ν are connected by the two equations $A_1 \quad \lambda B_1 + \mu C_1 + \nu D_1 = 0$, $\lambda^2 + \mu^2 + \nu^2 - 2\lambda\mu\nu = 1$ 77

2504. O is the centre of the circumscribing circle of a triangle ABC; PK, PL, PM, drawn parallel to CO, BO, AO respectively, meet AB, AC, BC in K, L, M; if KLM be a straight line, then the locus of P will be the conic
$$\frac{a}{\alpha}\cos A\cos(B-C) + \frac{b}{\beta}\cos B\cos(C-A) + \frac{c}{\gamma}\cos C\cos(A-B) = 0$$ 46

2507. Three circles pass through a point P; and PD, PE, PF are their diameters; if A, B, C be their other intersections, pror; that the three triads of points EAF, FBD, DCE are collinea and if a circle pass through the points P, D, E, F, the triad ABC is also collinear 61

2522. Prove (1) that the perpendiculars of a circular triangle have a common radical axis; and (2) that if the perpendiculars from the pairs of vertices of one circular triangle on the sides of another meet in a point, then *vice versâ*. (*Def.*—A, B, C, being circles, a circle coaxal with A, B, and orthogonal to C, is called the perpendicular from AB on C.)........................ 42

2524. Show that the equation to the circle circumscribing the triangle formed by the three straight lines $ax + by + c = 0$, $a'x...... = 0$, $a''x...... = 0$ may be thrown into the form
$$(ab'c'')(x^2+y^2) + \Sigma\frac{(a)^2+(b)^2}{(c)}(ax+by+c) = 0,$$
where $(ab'c'')$ is the determinant whose constituents are the coefficients in the three linear equations, and (a) (b) (c) (a') ......its first minors 39

2530. Trace the curve $\frac{1}{\sqrt{z}} + \frac{1}{\sqrt{(x+iy)}} + \frac{1}{\sqrt{(x-iy)}} = 0$, where the coordinates x, y, z are the perpendicular distances of the current point P from the sides of an equilateral triangle, the coordinates being positive for a point within the triangle...... 38

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When x becomes very large, the number of primes between $x-a$ and $x+a$ approaches the limit $\dfrac{2a}{\gamma+\log x}$, where $\gamma=0{\cdot}57721$......(Euler's constant) 85

A point O is taken within a triangle ABC, and straight lines AOD, BOE, COF, are drawn through O, meeting the sides in D, E, F; show that the average area of the triangle DEF, in parts of the area of the triangle ABC, is $10-\pi^2$ 49

If the diagonals of a convex quadrilateral divide each other in the ratios $\lambda : 1-\lambda$ and $\mu : 1-\mu$, the chance that the line passing through two points taken at random within the quadrilateral will meet both the diagonals at points within the quadrilateral is $\frac{1}{3}\left\{1+2(\lambda+\mu)-2(\lambda^2+\mu^2)\right\}$.
Thus, for a parallelogram the chance is $\frac{2}{3}$........................ 37

3. 1. Given any closed convex boundary of length L, and another of length l lying wholly *inside* the former; the probability that a line drawn at random in the plane and meeting L shall also meet l, is l : L. (N.B.—If the boundaries are not convex, this expression still holds good, if L denote, not the length of the boundary, but that of a string tightly wrapped round it; likewise for l.)
2. If l be the length of a closed convex boundary lying wholly *outside* L, X the length of an endless band tightly enveloping the two boundaries and crossing between them, and Y the length of another endless band also enveloping both but not crossing between them, the probability that a random line meeting L shall also meet l, is $(X-Y)$: L.
3. If the boundary l intersects L; then, if Y be the length of an endless band tightly enveloping both, the probability is $(L+l-Y)$: L 75

9. Solve the simultaneous equations
$yz+x=14,\quad zx+y=11,\quad zy+z=10$ 66

Show that the surface $y^2z^2+z^2x^2+x^2y^2-2xyz=0$ meets the sphere $x^2+y^2+z^2=1$ in four circles; and explain in a general manner the form of the curve of intersection of the surface by any other sphere having the same centre, and thence the form of the surface itself (being a particular case of Steiner's surface, and which by the homographic transformations $w^{-1}x$, $w^{-1}y$, $w^{-1}z$, for x, y, z gives $y^2z^2+z^2x^2+x^2y^2-2wxyz=0$, the general equation of Steiner's surface) 55

Rays being drawn from any point on a conic; the segments intercepted thereon between the conic and a fixed tangent can all be projected upon a given line, from another point on the conic, into segments of equal length 80

7. Show (1) that the locus of the centre of an ellipse of constant area touching the three sides of a triangle is a cubic hyperbola; and, supposing the centres to be equably distributed over every possible position within the triangle, show (2) that the average area of all the inscribed ellipses, in parts of the area of the triangle, is $\frac{4}{105}\pi^2$; moreover, on the same

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supposition of equable distribution of centres, show (3) that the average area of the triangle formed by joining the points of contact of the inscribed ellipses, in parts of the area of the triangle, is $10-\pi^2$ 98

2558. Mr. Henry Goodwin published in 1818 a table, in which all proper fractions (reduced to their lowest terms) in which the denominator did not exceed 100, nor the numerator 50, were arranged in order of magnitude. He observed the following property, a *general* proof of which is required. Let any three consecutive fractions be $\frac{N_1}{D_1}$, $\frac{N_2}{D_2}$, $\frac{N_3}{D_3}$, then $\frac{N_1+N_3}{D_1+D_3} = \frac{N_2}{D_2}$ 93

2559. Four quadrics pass each through three of four arbitrary lines in space; show that the four poles with respect to them of any arbitrary plane are coplanar 62

2562. Show that the equation of a Cartesian oval, referred to the triple focus as origin, is

$$\left\{x^2+y^2-(\beta\gamma+\gamma\alpha+\alpha\beta)\right\}^2+4\alpha\beta\gamma\left\{2x-(\alpha+\beta+\gamma)\right\}=0,$$

where α, β, γ are the distances of the three single foci from the triple focus 85

2569. If through a point O there be drawn three lines parallel respectively to the sides of a given triangle, the six points in which these lines meet the sides lie on a conic. Find its equation referred to the triangle. (See Question 2489.) 78

2573. The envelope of a variable circle having for its diameter the double ordinate of a rectangular cubic is a Cartesian.

[The expression "a rectangular cubic" is used to express a cubic with three real asymptotes, having a diameter at right angles to one of the asymptotes and at an angle of 45° to each of the other two asymptotes, viz., the equation of such a cubic is $xy^2 = x^3+bx^2+cx+d$.] 73

2575. Given two surfaces of the second degree, there exists *in general** a set of Cartesian axes, whose directions are those of conjugate diameters in every one of the surfaces of the second degree passing through the intersection* of the two surfaces given. The directions will be those of the three paraboloids* of the system of surfaces. The locus of centres of the system lies on the intersection of three cylinders of the second degree whose axes are parallel to those of the paraboloids of the system 53

* Real or imaginary: the reality of the reductions is another matter.

2576. If in Question 2537 the points D, E be taken arbitrarily, instead of the point O, show that the average area of the triangle DEF, in parts of the area of ABC, is $\frac{1}{16}(\pi^2-8)$ 52

2583. Prove that the expression

$$\begin{aligned}&\left\{(a-b)^2(c-d)^2+(a-c)^2(b-d)^2+(a-d)^2(b-c)^2\right\}^3\\&-8\left\{(a-b)^3(b-c)^3(c-d)^3(d-a)^3+(a-b)^3(b-d)^3(d-c)^3(c-a)^3\right.\\&\qquad\left.+(c-b)^3(b-d)^3(d-a)^3(a-c)^3\right\}\\&-24(a-b)^2(a-c)^2(a-d)^2(b-c)^2(b-d)^2(c-d)^2\end{aligned}$$

vanishes identically 112

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2590. It is required to verify Professor Kummer's theorem that "if a quartic surface is such that every section by a plane through a certain fixed point is a pair of conics, the surface is a pair of quadric surfaces (except only in the case where it is a quartic cone having its vertex at the fixed point)." ... 101

2592. If a finite straight line be marked at random in $p-1$ points and broken up at those points, (1) the chance that none of the pieces is less than one-mth of the whole ($m>p$) is $\left(1-\frac{p}{m}\right)^{p-1}$; (2) the chance that none of the pieces is greater than one-nth of the whole ($n<p$) is

$$\left(\frac{p}{n}-1\right)^{p-1}-p\left(\frac{p-1}{n}-1\right)^{p-1}+\frac{p(p-1)}{2}\left(\frac{p-2}{n}-1\right)^{p-1}-\dots$$

to r terms, r being the integer next $> p-n$; (3) the chance that none of the pieces is less than one-mth and none greater than one-nth is

$$\left(1-\frac{p}{m}\right)^{p-1}-p\left(1-\frac{p-1}{m}-\frac{1}{n}\right)^{p-1}+\frac{p(p-1)}{2}1\ \frac{p-2}{m}-\frac{2}{n}\Big)^{p-1}-\dots$$

to r terms, where r is the integer next $>n\frac{m-p}{m-n}$; provided that $\frac{1}{m}+\frac{p-1}{n}>1$ and $\frac{1}{n}+\frac{p-1}{m}<1$, otherwise one of the chances includes the other and has been already determined............ 103

2593. A cylindrical tube, closed at its lower end, and having an air-tight piston capable of moving freely in the upper part of it, rests vertically upon a solid pier. Above the piston there is a vacuum: below it there is air which at the time t is destitute of velocity, but of which the density varies as ϵ^{x}; where x is the distance of a stratum from the base of the cylinder. A constant pressure equal to the pressure on the piston of the air below it at the time t is exerted upon the piston downwards. Explain the subsequent motion, the air and piston being supposed without weight 84

2594. Let it be required to deal out ab cards n times into a sub-packs of b cards each, and so to arrange the sub-packs after each deal, that after the nth deal and arrangement of the sub-packs any selected card may stand in the rth place from the top of the whole pack 89

2601. Points P, Q, R, are taken at random on the sides of a triangle ABC; show that the average area of the triangle formed by the intersections of AP, BQ, CR, is $10-\pi^2$ 86

Unsolved Questions.

2392. (Proposed by Professor CAYLEY.)—It is required, by a real or imaginary linear transformation, to express the equation of a *given* cubic curve in the form

$$xy-z^2 = \surd\left\{(z^2-x^2)(z^2-k^2x^2)\right\}.$$

2394. (Proposed by Professor CREMONA.)—Les plans qui coupent en quatre points harmoniques une courbe gauche de 4^e ordre et 2^e espèce (courbe gauche unicursale de 4^e ordre, sans points doubles) enveloppent une surface de STEINER [see Crelle's Journal, vols. 63 and 64] pour laquelle la courbe donnée est asymptotique.

2446. (Proposed by W. K. CLIFFORD, B.A.)—PQ is a chord of a conic, equally inclined to the axis with the tangent at P. Any circle through PQ cuts the conic in RS. Show that the harmonic conjugate of RS relative to P lies on the straight line joining Q to the other extremity of the diameter through P. Hence show by inversion that, if chords be drawn to a circular cubic through the point where the asymptote cuts the curve, the locus of their middle points is a circle through the double point.

2455. (Proposed by the Rev. R. TOWNSEND, F.R.S.)—If a, b, c be the three sides of a spherical triangle, and k the radius of its polar circle, prove the formula

$$\tan^2 k = \frac{(\cos b\cos c\sec a-1)(\cos c\cos a\sec b-1)(\cos a\cos b\sec c-1)}{4\sin s\sin(s-a)\sin(s-b)\sin(s-c)}.$$

2505. (Proposed by the late Rev. R. H. WRIGHT, M.A.)—Prove that

$$\frac{(1+\frac{1}{4}x)^2}{(1+\frac{1}{3}x)(1+\frac{1}{5}x)}\cdot\frac{(1+\frac{1}{8}x)^2}{(1+\frac{1}{7}x)(1+\frac{1}{9}x)}\cdot\text{ad inf.} = \frac{2^{-\frac{3}{2}x}\,\Gamma(x+2)}{\Gamma(\frac{1}{2}x+1)\,.\,\Gamma(\frac{1}{4}x+1)}.$$

2531. (Proposed by Professor SYLVESTER.)—1. Let E denote the event of three points forming the apices of an acute angle triangle. Let p be the probability of E for three points taken anywhere inside a given triangle A; t the probability of E when the three points lie respectively anywhere upon the three sides of A; q, r, s the several probabilities of E when one point is taken at an angle of A, a second anywhere in the side opposite that angle, and a third anywhere inside A. Prove that $p = \frac{1}{5}q+\frac{1}{5}r+\frac{1}{5}s+\frac{2}{5}t$.

2. Obtain an analogous theorem for decomposing the probability of E when the original range of the three points is a tetrahedron.

3. Also give the corresponding theorems of decomposition when E denotes the event of a group of n points satisfying any intrinsic condition of form, and the original range of the group is a triangle, parallelogram, tetrahedron, or parallelepiped.

2555. (Proposed by A. DE MORGAN, F.R.A.S.)—The following is a theorem of which an elementary proof is desired. It was known before I gave it in a totally different form in a communication (April, 1867) to the Mathematical Society on the *conic octagram;* and the present form is as distinct from the other two as they are from one another. If I, II, III, IV be the consecutive chord-lines of one tetragon inscribed in a conic, and 1, 2, 3, 4 of another; the eight points of intersection of I with 2 and 4, II with 1 and 3, III with 2 and 4, IV with 1 and 3, lie in one conic section. A proof is especially asked for when the first conic is a pair of straight lines. There is, of course, another set of eight points in another conic, when the pairs 13, 24 are interchanged in the enunciation.

2564. (Proposed by M. Collins, B.A.)—A being a curve whose equation is given in the usual Cartesian rectangular coordinates, B the evolute of A, and C the evolute of B; required a general differential expression for the radius of curvature of C, on the usual supposition of dx being taken constant, and likewise on the supposition of dx^2+dy^2 $(=dz^2)$ being taken constant.

2565. (Proposed by M. W. Crofton, B.A.)—1. A convex boundary of any form, of length L, encloses an area Ω. If two straight lines are drawn at random to intersect it, the probability of their intersection lying within it is $p=2\pi\Omega L^{-2}$.

2. The probability of their intersection lying within any given area ω, which is enclosed within Ω, is $p=2\pi\omega L^{-2}$. [Mr. Crofton remarks that an interesting but much more difficult problem (a general solution to which is probably impracticable) is to find the chance of their intersection lying on a given area ω, *external* to Ω.]

3. If an infinity of random lines are drawn across the given area Ω, their intersections form an assemblage of points covering the plane, the density of which is clearly uniform *within* Ω. Show that, at any *external* point P, the density varies as $\theta-\sin\theta$, where θ is the angle Ω subtends at P.

4. If Ω be any plane area, enclosed by a convex boundary of length L, and θ be the angle it subtends at any external point $P(x, y)$, prove that $\iint(\theta-\sin\theta)\,dx\,dy=\frac{1}{2}L^2-\pi\Omega$, the integral extending over the whole external surface of the plane. [We shall be glad to receive a separate investigation of this theorem, independent of that which follows at once from the preceding theory.]

2574. (Proposed by W. S. B. Woolhouse, F.R.A.S.) — A given straight line is divided at random into n portions, and these are arranged in the order of magnitude; determine the average value of the portion which stands the mth in order.

2580. (Proposed by A. W. Panton, B.A.)—1. If F and F′ are the foci of an oval of Cassini, and C its centre; prove the following construction for the second pair of foci. The circle through F or F′ and the two points where any line through C meets either oval (the curve consisting of two distinct ovals) cuts the axis in one of the required points.

2. If S and S′ be the foci thus found, and P any point on the curve, prove that PS . PS′ is proportional to PC^2.

2597. (Proposed by W. H. H. Hudson, M.A.) — A right cone, whose weight may be neglected, is suspended from a point in its rim; it contains as much fluid as it can: prove that the whole pressure upon its surface is $\frac{1}{3}\pi\rho g h^3\,\dfrac{\sin\alpha\cos\theta}{\cos^2\alpha}\left\{\dfrac{\cos(\theta+\alpha)}{\cos(\theta-\alpha)}\right\}^{\frac{3}{2}}$, where h and 2α

are the height and vertical angle of the cone, and θ is determined from $3 \sin 2\theta = 4 \sin 2(\theta - \alpha)$.

2612. (Proposed by the Rev. J. WOLSTENHOLME, M.A.)—If three points P, Q, R be taken at random, one on each side of a triangle ABC, show that the respective chances of the triangle PQR being (1) greater than one-fourth, and (2) greater than one-half, the area of the triangle ABC, are $p_1 = \frac{1}{4} + \frac{1}{2} \log 2$, $p_2 = \frac{1}{2} + \frac{1}{12}\pi - \log 2$.

2616. (Proposed by C. W. MERRIFIELD, F.R.S.)—Let two intersecting tetrahedra have all their edges bisected by the same system of Cartesian axes, each axis through two opposite edges of each tetrahedron; then the solid about the origin has the origin for its centre of figure.

2618. (Proposed by the Rev. N. M. FERRERS, M.A.)—A pack of n different cards is laid, face downwards, on a table. A person names a certain card; that and all the cards above it are shown to him and removed; he names another, and the process is repeated: prove that the chance of his naming the top card during the operation is

$$\frac{1}{\lfloor\underline{2}} - \frac{1}{\lfloor\underline{3}} + \frac{1}{\lfloor\underline{4}} - \dots + \frac{(-1)^n}{\lfloor\underline{n}}.$$

2619. (Proposed by the EDITOR.)—Trace the form, and find the equation, length, and area of the first negative focal pedal of the ellipse, and deduce therefrom the same for the parabola. (See Solution of Question 1390, *Reprint*, Vol. I., p. 23.)

2621. (Proposed by the Rev. M. M. U. WILKINSON.)—If four points be taken at random on the surface of a sphere, show that the chance of their all lying on some one hemisphere is $\frac{7}{8}$.

2633. (Proposed by Professor TAIT.)—Show that the greatest amount of mechanical effect which can be obtained from a system of equal and similar masses (whose specific heat does not vary with their temperature) is proportional to the excess of the arithmetic over the geometric mean of their absolute temperatures.

2634. (Proposed by A. CRUM BROWN, D.Sc.)—$4m + 2n$ separate strings, having each a black and a white end, are taken. $4m$ of these are united in groups of four by their white ends, and the white ends of all the others are to be left free. In how many ways may the black ends of the system be united two and two so as to form a continuous aggregate? It is of course contemplated that two black ends belonging to the same group of four may be united. This gives rise to the further question:—Divide the above arrangements into classes according to the number of groups of four, each of which has two of its black ends united.

MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

2475. (Proposed by the EDITOR.)—Trace the curve whose equation in triangular coordinates (x, y, z) is

$$\left(\frac{1-x}{x}\right)^{x(1-x)}\left(\frac{1-y}{y}\right)^{y(1-y)}\left(\frac{1-z}{z}\right)^{z(1-z)}=\epsilon^{p};$$

that is, the curve which is the locus of P in Question 2359. (*Reprint*, Vol. VIII., p. 53.)

Solution by W. S. B. WOOLHOUSE, F.R.A.S.

As the triangular coordinates are not altered in value by orthographic projection, no generality will be lost if we consider the question in relation to an equilateral triangle. Also if the perpendicular of the equilateral triangle of reference be unity, the triangular coordinates will become the same as trilinear. And as the proposed equation is symmetrical, the form of the curve will be symmetrical with respect to the equal sides of the triangle of reference. If we put

$$\phi(x)=x(1-x)\log\frac{1-x}{x},$$

the logarithm of the proposed equation is

$$\phi(x)+\phi(y)+\phi(z)=p \;\ldots\ldots\; (a).$$

The curve may be effectually traced, and its properties practically developed, by first tabulating the function ϕx for the several values of x, and a specimen of such table is here given. It will be observed that

$$\phi(1-x)=-\phi(x),$$

and that the function is negative for the values of x given in the right-hand column. The numerical value of ϕx is most conveniently obtained by using common logarithms and employing the formula

$$\log\phi(x)=\log\left\{\log(1-x)-\log x\right\}+\left\{\log(1-x)+\log x\right\}+0{\cdot}3622156.$$

x	$\phi(x)$	x	x	$\phi(x)$	x	x	$\phi(x)$	x
·00	+ ·00000 −	1·00	·17	+ ·22373 −	·83	·34	+ ·14884 −	·66
·01	·04549	·99	·18	·22381	·82	·35	·14083	·65
·02	·07628	·98	·19	·22316	·81	·36	·13256	·64
·03	·10115	·97	·20	·22181	·80	·37	·12406	·63
·04	·12204	·96	·21	·21981	·79	·38	·11534	·62
·05	·13986	·95	·22	·21719	·78	·39	·10642	·61
·06	·15519	·94	·23	·21399	·77	·40	·09731	·60
·07	·16839	·93	·24	·21025	·76	·41	·08804	·59
·08	·17976	·92	·25	·20599	·75	·42	·07863	·58
·09	·18949	·91	·26	·20124	·74	·43	·06908	·57
·10	·19775	·90	·27	·19604	·73	·44	·05942	·56
·11	·20468	·89	·28	·19040	·72	·45	·04967	·55
·12	·21040	·88	·29	·18436	·71	·46	·03983	·54
·13	·21500	·87	·30	·17793	·70	·47	·02993	·53
·14	·21856	·86	·31	·17114	·69	·48	·01998	·52
·15	·22116	·85	·32	·16402	·68	·49	·01000	·51
·16	+ ·22287 −	·84	·33	+ ·15658 −	·67	·50	+ ·00000 −	·50

As an example of the calculation of positions generally, I have taken the case of $p = \cdot 40$, and by means of the table of $\phi(x)$ and equation (a) deduced the following series of points in this particular curve, the perpendicular of the triangle being 1.

$p = \cdot 40.$			Rectangular Coordinate $y' = \frac{y \backsim z}{\sqrt{3}}.$
x	y	z	
·19	·33127	·47873	± ·08514
·20	·30081	·49919	·11454
·22	·26324	·51676	·14637
·24	·23916	·52084	·16263
·26	·22229	·51771	·17056
·28	·20974	·51026	·17350
·30	·20032	·49968	·17284
·32	·19327	·48673	·16943
·34	·18792	·47208	·16406
·36	·18413	·45587	·15689
·38	·18159	·43841	·14828
·40	·18028	·41972	·13824
·42	·18030	·39970	·12667
·44	·18177	·37823	·11343
·46	·18500	·35500	·09815
·48	·19046	·32954	·08030
·50	·20059	·29941	·05705
·51	·20948	·28052	± ·04101

The peculiarities of the curves for different values of p may, however, be more simply inferred from certain special points which we proceed to notice.

With respect to the curve, it is obvious that the three lines which bisect the opposite sides of the triangle are each of them diametral. We may first find the two points in which the curve meets one of these diameters; to effect which we have to make $y = z = \frac{1}{2}(1-x)$; so that the equation (α) for determining the two values of x is

$$2\phi(x) + \phi\left(\frac{1-x}{2}\right) = p.$$

With the use of the preceding table for $\phi(x)$, we hence easily calculate the following values according to different values of p, viz.:—

$p = {\cdot}10,\ x' = {\cdot}0211$ and ${\cdot}8800$,
$p = {\cdot}20,\ x' = {\cdot}0540$ „ ${\cdot}7715$,
$p = {\cdot}30,\ x' = {\cdot}1017$ „ ${\cdot}6583$,
$p = {\cdot}40,\ x' = {\cdot}1801$ „ ${\cdot}5209$,
$p = {\cdot}42,\ x' = {\cdot}2050$ „ ${\cdot}4848$,
$p = {\cdot}44,\ x' = {\cdot}2382$ „ ${\cdot}4405$,
$p = {\cdot}45,\ x' = {\cdot}2619$ „ ${\cdot}4114$,
$p = {\cdot}46,\ x' = {\cdot}3032$ „ ${\cdot}3646$,

and $p = \frac{2}{3}\log 2 = {\cdot}4621$ is the maximum value, and belongs to the centre of gravity of the triangle. By symmetry these values equally determine two points in each diameter, making six axial points of the curve.

We may next find the positions of the points which lie in a straight line drawn through the centre parallel to a side of the triangle. For points on a line so drawn the following values of p for assumed values of z are obtained by direct calculation:—

$3x = 1$ $3z =$	$p =$	diff.	$3x = 1$ $3z =$	$p =$	diff.
0·00	·00000		0·50	·37757	
·05	·08003	+8003	·55	·39435	+1678
·10	·13562	5559	·60	·40907	1472
·15	·18150	4588	·65	·42183	1276
·20	·22093	3943	·70	·43272	1089
·25	·25542	3449	·75	·44180	908
·30	·28590	3048	·80	·44916	736
·35	·31298	2708	·85	·45485	569
·40	·33710	2412	·90	·45888	403
·45	·35854	2144	·95	·46129	241
·50	·37757	+1903	1·00	·46210	+ 81

With the use of a table of this kind, extended to each hundredth of value of $3z$, it is easy to directly deduce the value of $3z$ for any given value of p. Then if ρ denote the distance of the point from the centre, the perpendicular of the triangle being unity, we shall have

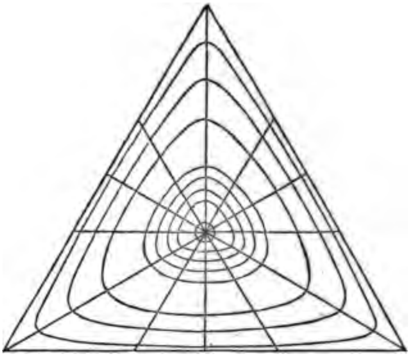
$$\rho = \frac{2}{3\sqrt{3}}(1-3z) = [9{\cdot}58535]\,(1-3z).$$

The three pairs of points situated on the three lines drawn respectively parallel to the sides are all of them at this same distance from the centre, and are evidently the corners of a regular hexagon. Also if these six points be taken with the six before found, at the distances $x' \backsim \frac{1}{3}$ from the centre, we shall have twelve points of the curve distributed at equal angles round the centre, and these will obviously present a sufficient indication of the particular form of the curve.

If we take as examples the probabilities before assumed, the calculated distances of the twelve special points from the centre are here tabulated.

Prob. p.	Central distances of points		
	on parallel.	towards an angle.	towards a side.
·00	·3849	·6667	·3333
·10	·3591	·5467	·3122
·20	·3184	·4382	·2790
·30	·2596	·3250	·2315
·40	·1661	·1875	·1532
·42	·1375	·1514	·1280
·44	·1002	·1072	·0950
·45	·0744	·0776	·0712
·46	·0306	·0307	·0294

By drawing the diameters and parallels, and laying down the sets of twelve points according to these distances, and then tracing a continuous line through each set of points, the several curves are delineated as shown in the annexed diagram.



It is evident that a similar process is applicable to any curve the trilinear equation of which is symmetrical. In the present case the curve is always convex outwards. For the higher probabilities, near the centre, it is nearly circular; but as the probability becomes less and the curve expands, it becomes partially flattened towards the three sides of the triangle, as if it were in imitation of those sides which, in fact, the curve ultimately becomes when p vanishes.

We may add that it is hence easy to conceive what will become of the several parts of the figure when it is orthogonally projected so as to transform the equilateral triangle into another of any given form.

NOTE ON QUESTION 2471. BY PROFESSOR CAYLEY.

In the singularly beautiful solution which Mr. WOOLHOUSE has given of this question, (see *Reprint*, Vol. VIII., p. 100) it is important to note what is the analytical problem solved, and how the solution is obtained. Considering a plane area bounded by any closed convex curve, and in it three points P, P′, P″, Mr. Woolhouse in vestigates the average area of the triangle PP′P″, viz., this depends on the sextuple integral

$$\int \pm \left\{x'y' - x''y' + x''y - xy'' + xy' - x'y\right\} dx\, dy\, dx' dy' dx'' dy'',$$

where the sign $\pm$ has to be taken so that $\pm \{\,\}$ shall be positive, and where the integration in respect to each set of coordinates extends over the entire closed area ; the difficulty is as to the mode of dealing with the discontinuous sign. It is remarked that the integral is

$$=6\int \pm \left\{x'y'' - x''y' + x''y - xy'' + xy' - x'y\right\} dx\, dy\, dx' dy' dx'' dy'' ;$$

the variables in this last expression being restricted in such wise that x, x'', x' are in the order of increasing magnitude ; the term $\pm \{\,\}$ is of the form

$$\pm\,(x'-x)\,(y''-\beta)$$

where β is independent of y, and where (as is easily seen) if v'', u'', be the upper and lower ordinate corresponding to the abscissa x'', then β lies between the values u'' and v''. But $x'-x$ is positive, hence the sign $\pm$ must be so taken that $\pm\,(y''-\beta)$ shall be positive, that is, from $y''=u''$ to $y''=\beta$ the sign is $-$, and from $y''=\beta$ to $y''=v''$ the sign is $+$.

Whence for the integration in regard of y'' we have

$$\int \pm\,(y''-\beta)\,dy'' = \int_{\beta}^{b''} +(y''-\beta)\,dy'' + \int_{u'}^{\beta} -(y''-\beta)\,dy''$$

$$= \tfrac{1}{2}\,(v''-\beta)^2 + \tfrac{1}{2}\,(\beta-u'')^2 ;$$

and the discontinuous sign $\pm$ is thus got rid of. The remaining integrations are then effected in the order x'', y', y, x', x, the limits being for x'' from x to x', for y' from u' to v', and for y from u to v (if the upper and lower ordinates corresponding to the abscissa x and x' are v, u and v', u' respectively) and finally for x' from x to the maximum abscissa, and for x from the minimum to the maximum abscissa. The final result involves only single definite integrals between the extreme values of x, the functions under the integral sign containing indefinite integrations from the same arbitrary inferior limit, say $x=0$; the form of the result (previous to its simplification by taking the axes to be principal axes through the centre of gravity of the area) is however somewhat complicated; and it would not be easy to show *a posteriori*, that the value is invariantive, that is, independent of the position of the axes : that this is so is of course apparent from the original form of the integral.

2491. (Proposed by the Rev. R. Harley, F.R.S.)—Solve, by definite integration, the equation

$$(x^3-1)\frac{d^3y}{dx^3} - 3ax^2\frac{d^2y}{dx^2} + 3a\,(a+1)\,x\frac{dy}{dx} - a\,(a+1)\,(a+2)\,y = 0.$$

Solution by the Rev. J. L. Kitchin.

1. Put $x = \epsilon^t$, and D for $\dfrac{d}{dt}$; then the equation becomes

$$\epsilon^{3\theta}(\mathrm{D}-a)\,(\mathrm{D}-a-1)\,(\mathrm{D}-a-2)\,y - \mathrm{D}\,(\mathrm{D}-1)\,(\mathrm{D}-2)\,y = 0,$$

therefore $y - \frac{D-a-3}{D}\epsilon^{\theta} \cdot \frac{D-a-3}{D}\epsilon^{\theta} \cdot \frac{D-a-3}{D}\epsilon^{\theta} \cdot y = 0,$

or $$\left\{1 - \left(\frac{D-a-3}{D}\epsilon^{\theta}\right)^{3}\right\} y = 0.$$

The solution of this depends on the roots of $1-\rho^3 = 0$. Let these roots be $1, \omega_1, \omega_2$; then

$$\frac{1}{1-\rho^3} = \frac{1}{3(1-\rho)} - \frac{\omega_1}{\omega_2(1-\omega_1)^2} \cdot \frac{1}{\omega_1-\rho} - \frac{\omega_2}{\omega_1(1-\omega_2)^2} \cdot \frac{1}{\omega_2-\rho}.$$

So that if y_1, &c., denote the values of y obtained from these forms separately their sum will be the value of y.

Now $$y_1 - \frac{D-a-3}{D}\epsilon^{\theta} \cdot y_1 = 0 \text{ gives } y_1 = C_1(x-1)^{a+2}$$

$$\omega_1 y_2 - \frac{D-a-3}{D}\epsilon^{\theta} \cdot y_2 = 0 \text{ gives } y_2 = C_2(x-\omega_1)^{a+2}$$

$$\omega_2 y_3 - \frac{D-a-3}{D}\epsilon^{\theta} \cdot y_3 = 0 \text{ gives } y_3 = C_3(x-\omega_2)^{a+2}$$

$$\therefore\ y = \frac{C_1}{3}(x-1)^{a+2} - \frac{\omega_1 C_2 (x-\omega_1)^{a+2}}{\omega_2(1-\omega_1)^2} - \frac{\omega_2 C_3 (x-\omega_2)^{a+2}}{\omega_1(1-\omega_2)^2}.$$

2. This equation forms one of a class whose symbolic expression is

$$y - \frac{\left\{(D-a-n)\,\epsilon^{\theta}\right\}^{n}}{[D]^{n}}\, y = 0,$$

where the lower line is not exponential.

Its general form, not expressed in symbols of operation, is

$$(x^n-1)\frac{d^n y}{dx^n} - \frac{n}{1}x^{n-1}[a]^1\frac{d^{n-1}y}{dx^{n-1}} + \frac{n(n-1)}{1.2}x^{n-2}[a]^2\frac{d^{n-2}y}{dx^{n-2}}$$

$$-\frac{n(n-1)(n-2)}{1.2.3}x^{n-3}[a]^3\frac{d^{n-3}y}{dx^{n-3}} + \ldots + (-1)^n[a]^n = 0,$$

where $[a]^n$ denotes $a(a+1)(a+2)\ldots(a+n-1)$.

It is plain also that this question can be solved, when, instead of being equated to zero, it is equal to a function of x, as X. The equation will then split up into n equations of the form $\frac{dy}{dx} + Py = \phi$, where P and Q are both functions of x; and its general integral can be found whenever these n equations can be integrated in finite terms.

[The Proposer remarks, that the first member of the given equation may be decomposed into three linear differential factors of the first order and degree, and that the equation may be thus put under the form

$$\left[(x-1)\frac{d}{dx} - a\right]\left[(x-\omega^2)\frac{d}{dx} - (a+1)\right]\left[(x-\omega)\frac{d}{dx} - (a+2)\right] y = 0,$$

where ω is an unreal cube root of unity. From this, by successive integrations, he deduces the following results, viz.,

$$\left[(x-\omega^3)\frac{d}{dx}-(a+1)\right]\left[(x-\omega)\frac{d}{dx}-(a+2)\right]y = C(x-1)^a,$$

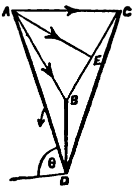
$$\left[(x-\omega)\frac{d}{dx}-(a+2)\right]y = C'(x-1)^{a+1}+C''(x-\omega^2)^{a+1},$$

$$y = C_1(x-1)^{a+2}+C_2(x-\omega)^{a+2}+C_3(x-\omega^2)^{a+2}.$$

The final result agrees with Mr. Kitchin's, though it is here expressed in a somewhat simpler form.]

2400. (Proposed by T. T. WILKINSON, F.R.A.S.)—Three equal and heavy rods, in the position of the three edges of an inverted pyramid, are in equilibrium under the following circumstances:—their upper extremities are connected by strings of equal lengths, and their lower extremities are attached to a hinge about which the rods may move freely in all directions. Show that the increase of tension of the strings, corresponding to a given increase of their lengths, varies as cosec³ θ, where θ is the inclination of each of the rods to the horizontal plane.

Solution by the PROPOSER; J. DALE; *and others.*



Let ABC be the equilateral triangle formed by the three strings; AD, BD, CD, the three rods; and AE a perpendicular to BC. Put $CA=a$, $AD=2b$, T=the tension of a string; and W = weight of each rod. Then $\angle BAE=30°$; and $2T\cos 30°$=resultant of the tension of AC and AB; which evidently acts in the line AE. This line will also be in the same plane with AD; hence, taking moments round the point D, we have

$$Wb\cos\theta = 2T\cos 30°\, 2b\sin\theta = 0, \therefore T = \frac{W}{2\sqrt{3}}\cot\theta.$$

But $AE = a\cos 30°$, and $\frac{2}{3}AE = 2b\cos\theta$; $\therefore a = 2\sqrt{3}\,.\,b\cos\theta$.

Hence $$\frac{dT}{da} = \frac{dT}{d\theta}\cdot\frac{d\theta}{da} = \frac{W}{12b}\operatorname{cosec}^3\theta,$$ which proves the theorem.

NOTE ON THE SOLUTION OF QUESTION 2452. BY Z.

In the solutions of Question 2452, given on p. 61 of Vol. VIII., of the *Reprint*, it appears to be assumed that if the point Q is not within the partial tetrahedron PABC, it must complete a convex solid PQABC. Passing over the consideration that if QPD happen to be in the same line, Q between P and D is within none of the four partial 4-edra; it depends upon the way in which we define a convex solid, whether Q external to PABC will always complete a convex solid ABCPQ, or not. The definition of a convex solid, according to Professor Sylvester, is evidently a solid which has no reentrant solid angle. But what if it has a reentrant solid edge? If on the opposite faces of a triangle *abc* we lay two 4-edra *pabc* and *qabc*, so that the faces *pab* and *qab* make a very acute angle, is such a solid *abcpq* convex, when it has the reentrant edge *ab*? If the affirmative is not admitted, the demonstration may have to be reconsidered; for Q external to the 4-edron ABCP will not complete a convex solid ABCPQ in the strict sense, unless the point Q is within one of the solid angles subtended by the three triangles ABP, BCP, CAP. If Q, outside the plane ABP, is in the solid angle subtended by it, a 6-edron is completed by the edges QA, QB, QP. If Q be in the plane ABP, and not in the produced edge PA or PB, a 5-edron is completed having the quadrilateral face APQB. But if Q is not in such a solid angle, but in the dihedral angle, *e.g.* of the planes ABP, BCP, it will be impossible to complete a solid ABCPQ, of which ABC shall be a triangular face, which shall not have the reentrant edge BP. We must draw either the three edges QP, QB, QA, or QP, QB, QC. In the first case QC will be drawable, *i.e.* a line through two angles of the solid ABCPQ, which neither pierces the solid nor is an edge of it: in the other case QA is drawable under the same conditions, a construction impossible in a strictly convex solid.

2478. (Proposed by J. J. WALKER, M.A.)—Deduce the first integrals

$$f+x\left(\frac{\mu}{r}-\frac{dy^2+dx^2}{dt^2}\right)+\frac{ydy+zdz}{dt}\cdot\frac{dx}{dt}=0,\ f'\ \&c.\ \&c.$$

(*Méc. Cél.* t. II., c. iii., §. 18) directly from the three equations of motion

$$(1)\ldots\frac{d^2x}{dt^2}+\frac{\mu x}{r^3}=0,\quad(2)\ldots\frac{d^2y}{dt^2}+\frac{\mu y}{r^3}=0,\quad(3)\ldots\frac{d^2z}{dt^2}+\frac{\mu z}{r^3}=0;$$

these first integrals being merely shown by LAPLACE to satisfy the last three differential equations.

Solution by the PROPOSER.

Multiplying (2) by $2x\frac{dy}{dt}$ and (3) by $2x\frac{dz}{dt}$ and adding, there results

$$2x\left(\frac{d^2y}{dt^2}\frac{dy}{dt}+\frac{d^2z}{dt^2}\frac{dz}{dt}\right)+\frac{2\mu x^2}{r^3}\left(y\frac{dy}{dt}+z\frac{dz}{dt}\right)=0\ \ldots\ldots\ldots(4).$$

Also we have identically

$$\frac{d}{dt}\left(\frac{\mu x}{r}\right)+\frac{\mu x}{r^3}\left(y\frac{dy}{dt}+z\frac{dz}{dt}\right)-\mu\frac{y^2+z^2}{r^3}\frac{dx}{dt}=0\ \ldots\ldots\ldots\ldots(5),$$

and $$\frac{d}{dt}\left(\frac{ydy+zdz}{dt}\cdot\frac{dx}{dt}\right)-\left(\frac{dy^2}{dt^2}+\frac{dz^2}{dt^2}\right)\frac{dx}{dt}-\left(y\frac{d^2y}{dt^2}+z\frac{d^2z}{dt^2}\right)\frac{dx}{dt}$$
$$-\frac{ydy+zdz}{dt}\frac{d^2x}{dt^2}=0,$$

which latter equation in virtue of (1) (2) (3) becomes

$$\frac{d}{dt}\left(\frac{ydy+zdz}{dt}\cdot\frac{dx}{dt}\right)-\left(\frac{dy^2}{dt^2}+\frac{dz^2}{dt^2}\right)\frac{dx}{dt}+\mu\frac{y^2+z^2}{r^3}\frac{dx}{dt}+\frac{\mu x}{r^3}\frac{ydy+zdz}{dt}=0\,..\,(6).$$

Subtracting (4) from the sum of (5) and (6)

$$\frac{d}{dt}\left(\frac{ydy+zdz}{dt}\cdot\frac{dx}{dt}+\frac{\mu x}{r}\right)-2x\left(\frac{d^2y}{dt^2}\frac{dy}{dt}+\frac{d^2z}{dt^2}\frac{dz}{dt}\right)-\left(\frac{dy^2}{dt^2}+\frac{dz^2}{dt^2}\right)\frac{dx}{dt}=0.$$

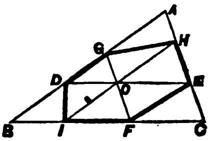
Integrating, f being the arbitrary constant,

$$f+\frac{ydy+zdz}{dt}\frac{dx}{dt}+\frac{\mu x}{r}-x\left(\frac{dy^2}{dt^2}+\frac{dz^2}{dt^2}\right)=0.$$

By interchanging x and y, x and z successively, we have two other analogous first integrals.

2489. (Proposed by S. WATSON.)—Through a point O, within a triangle ABC, parallels DE, FG, HI are drawn to BC, CA, AB, respectively; find the locus of O when the area of the hexagon DIFEHG is constant.

I. *Solution by* T. DOBSON, B.A.; W. H. LAVERTY; W. CHADWICK; *and others.*



The sum of the parallelograms (OA, OB, OC) at the vertices of the triangle is evidently constant, therefore the sum of the remaining three triangles (OIF, OEH, OGD) is also constant, and $= n^2\Delta$ say.

If (α, β, γ) be the trilinear coordinates of O, the base of the triangle OIF is

$$\text{IF} = \alpha\,(\cot \text{B} + \cot \text{C}) = \alpha \sin \text{A} \operatorname{cosec} \text{B} \operatorname{cosec} \text{C}.$$

Hence $\alpha^2 \sin^2 \text{A} + \beta^2 \sin^2 \text{B} + \gamma^2 \sin^2 \text{C} = 2n^2\Delta \,.\sin \text{A} \sin \text{B} \sin \text{C}$,
or, multiplying by 4R^2, $a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2 = n^2\,(2\Delta)^2$
is the equation to the locus required.

In triangular coordinates this takes the simple form

$$x^2 + y^2 + z^2 = n^2 \ldots\ldots\ldots\ldots\ldots\ldots (\alpha).$$

[If we put $k = \dfrac{n^2}{1-n^2} = \dfrac{2m-1}{2-2m}$, where $m\Delta$ is the area of the hexagon, the equation (α) may be written

$$x^2 + y^2 + z^2 - 2k\,(yz + zx + xy) = 0 \ldots\ldots\ldots (\beta),$$

which shows that the locus of O is an *ellipse*, having its centre at the centre of gravity of the triangle ABC. If the hexagon is three-fourths of the triangle, we have $m = \frac{3}{4}$, $k = 1$; and the equation (β) of the locus becomes

$$x^{\frac{1}{2}} + y^{\frac{1}{2}} + z^{\frac{1}{2}} = 0 \ldots\ldots\ldots\ldots (\gamma),$$

which is that of an ellipse inscribed in the triangle.]

II. *Solution by* R. TUCKER, M.A.

Let CI $= a-x = na$, then $n = \dfrac{a-x}{a}$; and also IH $= nc$, and CH $= nb$;

therefore $\mathrm{CF} = (nc-y)\dfrac{a}{c}$, $\mathrm{FO} = \dfrac{by}{c}$, $\mathrm{AG} = nc-y$, $\mathrm{AH} = b(1-n)$.

Then the difference between twice the triangle and twice the hexagon = a constant $= \lambda^2 \sin \mathrm{B}$ suppose; hence the equation to the locus of O is

$$xy \sin \mathrm{B} + (a-x)\,y \sin \mathrm{B} - \frac{aby^2}{c^2} \sin \mathrm{C} + \frac{bx}{a}\left(\frac{a-x}{a}\,c-y\right) \sin \mathrm{A} = \lambda^2 \sin \mathrm{B},$$

or
$$a^2y^2 + acxy + c^2x^2 - a^2cy - ac^2x + \lambda^2ac = 0,$$

the equation to an ellipse [possessing the properties stated in the note to the foregoing solution.]

2470. (Proposed by the Rev. J. BLISSARD.)—Prove that

$$\frac{n-1}{1}\cdot\frac{2^2}{3} - \frac{(n-1)\,(n-2)}{1\,.\,2}\cdot\frac{2^3}{4} + \&\mathrm{c.} = \frac{n+2}{n+1},\ (n\ \mathrm{even}).$$

Solution by SAMUEL ROBERTS, M.A.

We have $x\,(1-x)^{n-1} = x-(n-1)\,x^2 + \dfrac{(n-1)\,(n-2)}{1\,.\,2}\,x^3 - \&\mathrm{c.}$

Consequently, if N denote the left hand member of the equality in the question,

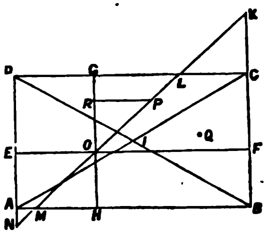
$$\mathrm{N} = 1 - \frac{1}{2}\int_0^2 x\,(1-x)^{n-1}\,dx = 1 + \frac{1}{n+1}\ (n\ \mathrm{even}).$$

If for the arguments 2^2, 2^3, &c., we write p^2, p^3, &c., the general expression is

$$\frac{p}{2} - \frac{1}{p}\left\{\frac{1-(1-p)^n\,(1+np)}{n\,(n+1)}\right\};\ \text{and for } p=2\ (n\ \text{odd}), \text{ we have } \mathrm{N}' = \frac{n-1}{n}.$$

2350. (Proposed by W. S. B. WOOLHOUSE, F.R.A.S.) — Determine the average area of all triangles drawn on the surface of a given rectangle which have one angle at a given point.

I. *Solution by* STEPHEN WATSON.



Let O be the given point in the rectangle ABCD, through which draw EF, GH, parallel respectively to AB, AD; and draw also NOK meeting AD, AB, CD, CB in N, M, L, K respectively. Take P any point in OL, and draw PR parallel to CD. Put AB $= a$, BC $= b$, AH $= a_1$, HB $= a_2$, AE $= b_1$, ED $= b_2$, RP $= x$, OR $= y$, and $\angle$ ROP $= \phi$. Then when Q lies in the triangles MBK, LCK, LDN, MAN, the sum of the areas of the triangle OPQ is respectively

$$\frac{1}{12xy}(a_2y+b_1x)^3\ldots.(1),\quad \frac{1}{12xy}(a_2y-b_2x)^3\ldots.(2),$$

$$\frac{1}{12xy}(a_1y+b_2x)^3\ldots.(3),\quad \frac{1}{12xy}(a_1y-b_1x)^3\ldots.(4).$$

For OP$=y\sec\phi$, distance of centroid of triangle MBK from MK $=$ $\frac{1}{3}$BK $\sin\phi$, and area of triangle MBK $=\frac{1}{2}$BK^2 $\tan\phi$; hence

$$\tfrac{1}{2}\text{OP}\,.\,\tfrac{1}{3}\text{BK}\sin\phi\,.\,\tfrac{1}{2}\text{BK}^2\tan\phi=\tfrac{1}{12}\,y\tan^2\phi\,(a_2\cot\phi+b_1)^3=\frac{1}{12xy}(a_2y+b_1x)^3,$$

which proves (1). Similarly (2), (3), (4) may be established. Since Q must always lie in the rectangle ABCD, (2) and (4) are *subtractive*; but when $x>\frac{a_2y}{b_2}$ (say $>x_2$), the triangle LCK is formed on the opposite side of C, is still subtractive, and (2) is *negative*; hence in this case (2) becomes *additive*. Similarly when $x>\frac{a_1y}{b_1}$ (say $>x_1$), (4) becomes additive. Consequently when Q takes every position in the rectangle ABCD, and P in OFCG, the sum of the areas of the triangle OPQ is

$$\int_0^{a_2}dy\left\{\int_0^{a_2}[(1)-(2)+(3)-(4)]\,dx+2\int_{x_2}^{a_2}(2)\,dx+2\int_{x_1}^{a_2}(4)\,dx\right\}$$

$$=\int_0^{b_2}dy\left\{\frac{y^2}{6}\left(a_1^3\log\frac{a_2b_1}{a_1y}+a_2^3\log\frac{b_2}{y}\right)+\tfrac{1}{8}aa_2^2(b_1^2+b_2^2)\right.$$
$$\left.-\tfrac{1}{4}aa_2(a_1-a_2)(b_1-b_2)\,y+\tfrac{13}{36}(a_1^3+a_2^3)\,y^2\right\}$$

$$=\tfrac{1}{18}a_1^3b_2^3\log\frac{a_2b_1}{a_1b_2}+\tfrac{13}{108}(a_1^3+a_2^3)\,b_2^3+\tfrac{1}{8}aa_2b_2\left\{ba_2b_1-a_1b_2(b_1-b_2)\right\}\ldots(5).$$

Under similar conditions, when P lies in the rectangles AHOE, OEDG, OFBH, the sum of the areas is, respectively,

$$\tfrac{1}{18}a_1^3b_2^3\log\frac{a_2b_1}{a_1b_2}+\tfrac{13}{108}(b_1^3+b_2^3)\,a_1^3+\tfrac{1}{8}ba_1b_1\left\{aa_2b_1+a_1b_2(a_1-a_2)\right\}\ldots.(6),$$

$$\tfrac{1}{18}a_1^3b_1^3\log\frac{a_2b_2}{a_1b_1}+\tfrac{13}{108}(b_1^3+b_2^3)\,a_1^3+\tfrac{1}{8}ba_1b_2\left\{aa_2b_2+a_1b_1(a_1-a_2)\right\}\ldots.(7),$$

$$\tfrac{1}{18}a_1^3b_1^3\log\frac{a_2b_2}{a_1b_1}+\tfrac{13}{108}(a_1^3+a_2^3)\,b_1^3+\tfrac{1}{8}aa_2b_1\left\{ba_2b_2+a_1b_1(b_1-b_2)\right\}\ldots.(8).$$

Therefore the required average is

$$\frac{1}{a^2b^2}\left\{(5)+(6)+(7)+(8)\right\}=\frac{1}{a^2b^2}\left\{\frac{a_1^3}{9}\left(b_1^3\log\frac{a_2b_2}{a_1b_1}+b_2^3\log\frac{a_2b_1}{a_1b_2}\right)\right.$$

$$\left.+\tfrac{13}{108}(3a_1^3+a_2^3)(b_1^3+b_2^3)+\tfrac{1}{4}aba_1a_2(b_1^2+b_2^2)+\tfrac{1}{4}bb_1b_2(a_1^2-aa_2)(a_1-a_2)\right\}\text{(9).}$$

The above is on the supposition that O lies above AC and below BD, that is in the triangle AID. When O lies in the triangle AIB the average is

$$\frac{1}{a^2b^2}\left\{\frac{b_1^3}{9}\left(a_1^3\log\frac{a_2b_2}{a_1b_1}+a_2^3\log\frac{a_1b_2}{a_2b_1}\right)+\tfrac{13}{108}(a_1^3+a_2^3)(3b_1^3+b_2^3)\right.$$

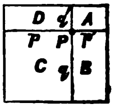
$$\left.+\tfrac{1}{4}abb_1b_2(a_1^2+a_2^2)+\tfrac{1}{4}aa_1a_2(b_1^2-bb_2)(b_1-b_2)\right\}\ldots\ldots\text{(10).}$$

When O lies in the triangle BIC we have merely to interchange a_1, a_2 and b_1, b_2 in (9); and when in the triangle CID the same interchange of letters in (10).

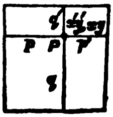
When O lies at I, $a_1=a_2=\frac{1}{2}a$, $b_1=b_2=\frac{1}{2}b$, and the average is $\frac{5}{108}ab$.

II. *Solution by the* Proposer.

If the diagram be orthographically projected upon another plane making any assigned angles with that of the given rectangle, it is evident that the area of the triangle will always have the same relative value when compared with the total area. We may therefore simplify the investigation by taking a square, the side of which is unity, as it can be projected into a rectangle or parallelogram of any proposed form.



Let P be the given point, and let it be taken as the origin, with the coordinate axes drawn parallel to the sides of the square, the segments of these axes, intercepted by the sides, being denoted by p, p', q, q', as marked in the annexed diagram. For the sake of brevity, the four coordinate spaces into which the square is divided are designated by the letters A, B, C, D, so as to enable us to distinguish the different localities in which the points are supposed to be taken. We shall also denote the sum of the areas of the triangles by (AA) when the two points Q, R both occupy the space A; by (AB) when one of those points is on A and the other on B; by (AC) when one point is on A and the other on C; and so on. The three cases here stated will determine all others by analogy.



Case I.—Let Q (xy) and R $(x'y')$ be both in A, with R to the left of Q, or $x'<x$.

1. When R is above the side PQ, $\Delta=\dfrac{xy'-x'y}{2}$, and the limits are $y'\left(y\dfrac{x'}{x}\ldots q'\right)$, $x'\,(o\ldots p')$, &c.

therefore
$$\int\Delta\,dy'=\frac{(xy'-x'y)^2}{4x}=\frac{(q'x-x'y)^2}{4x},$$

$$\iint\Delta\,dx'dy'=-\frac{(q'x-x'y)^3}{12xy}=\frac{q'^3x^3-(q'-y)^3x^3}{12xy},$$

$$=\frac{x^2}{12y}(3q'^2y-3q'y^2+y^3)=\frac{q'^2x^2}{4}-\frac{q'x^2y}{4}+$$

$$\iiint \Delta\, dy\, dx'\, dy' = q'^3x^2\left(\tfrac{1}{4}-\tfrac{1}{8}+\tfrac{1}{36}\right) = \tfrac{11}{72}q'^3x^2,$$

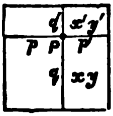
$$\iiiint \Delta\, dx\, dy\, dx'\, dy' = \tfrac{11}{216}p'^3q'^3 \ldots\ldots\ldots\ldots (1).$$

2. When R is below PQ, $\Delta = \frac{1}{2}(x'y-xy')$, and the limits are $y'\left(o\ldots y\dfrac{x'}{x}\right)$, $x'(o\ldots x)$, &c.

therefore $\int \Delta\, dy' = -\dfrac{(x'y-xy')^2}{4x} = \dfrac{x'^2y^2}{4x}$, $\iint \Delta\, dx'\, dy' = \dfrac{x'^3y^2}{12x} = \dfrac{x^2y^2}{12}$,

$$\iiiint \Delta\, dx\, dy\, dx'\, dy' = \frac{p'^3q'^3}{108} \ldots\ldots\ldots\ldots (2).$$

therefore $\qquad (AA) = p'^3q'^3\left(\tfrac{11}{216}+\tfrac{1}{108}\right) = \tfrac{13}{216}p'^3q'^3.$



Case II.—Let Q be in B and R in A; then, changing the sign of y, $\Delta = \frac{1}{2}(xy'+x'y)$, and y', x', y, x take full limits;

therefore $\int \Delta\, dy' = \dfrac{(xy'+x'y)^2}{4x} = \dfrac{(q'x+x'y)^2-x'^2y^2}{4x}$

$$= \frac{q'^2x^2+2q'xx'y}{4x} = \frac{q'^2x}{4}+\frac{q'x'y}{2},$$

$$\iint \Delta\, dx'\, dy' = \frac{q'^2xx'}{4}+\frac{q'yx'^2}{4} = \frac{p'q'^2}{4}x+\frac{p'^2q'}{4}y,$$

$$\iiint \Delta\, dy\, dx'\, dy' = \frac{p'q'^2}{4}xy+\frac{p'^2q'}{8}y^2 = \frac{p'q'^2}{4}qx+\frac{p'^2q'}{8}q^2,$$

$$\iiiint \Delta\, dx\, dy\, dx'\, dy' = \frac{p'q'^2}{8}qx^2+\frac{p'^2q'}{8}q^2x = \tfrac{1}{8}(p'^3q'^2q+p'^3q'q^2);$$

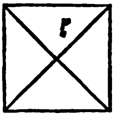
therefore $\qquad (AB) = \dfrac{p'^3}{8}qq'(q+q') - \dfrac{p'^3}{8}qq'.$

Hence, for triangles which lie to the right of P,

$$(AA)+(BB)+(AB) = p'^3\left\{\tfrac{13}{216}(q^3+q'^3)+\tfrac{1}{8}qq'(q+q')\right\}$$

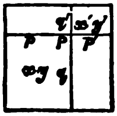
$$= p'^3\left\{\tfrac{13}{216}(q+q')^3-\tfrac{1}{18}qq'(q+q')\right\} = p'^3\left(\tfrac{13}{216}-\tfrac{1}{18}qq'\right),$$

which is therefore the form of expression when P is in the side of a rectangle.



Case III.—Let Q be in C and R in A; and suppose the fixed point P to be situated in the upper diagonal space as in the first of the annexed diagrams; then $pq' < p'q$, or $\dfrac{q'}{p'} < \dfrac{q}{p}$.

1. When R is above PQ, $\Delta = \dfrac{xy'-yx'}{2}$, and we have the limits $y'\left(y\dfrac{x'}{x}\ldots q'\right)$;



therefore $\qquad \int \Delta\, dy' = \dfrac{(xy'-yx')^2}{4x} = \dfrac{(q'x-yx')^2}{4x}.$

The other limits are as follows:—

$$\left.\begin{array}{l|l} \dfrac{y}{x} < \dfrac{q'}{p'} & x'(o\ldots p'),\quad y\left(o\ldots x\dfrac{q'}{p'}\right),\ x(o\ldots p)\ldots\ldots\ldots(1), \\ \dfrac{y}{x} > \dfrac{q'}{p'} & x'\left(o\ldots q'\dfrac{x}{y}\right),\ y\left(x\dfrac{q'}{p'}\ldots q\right),\ x(o\ldots p)\ldots\ldots\ldots(2); \end{array}\right.$$

therefore (1) $\iint \Delta\, dx'\, dy' = -\dfrac{(q'x-yx')^3}{12xy} = \dfrac{q'x^3-(q'x-yx')^3}{12xy}$

$$= q'x'\frac{q'x-yx'}{4} + \frac{y^3x'^3}{12xy} = p'q'\frac{q'x-p'y}{4} + \frac{p'^3y^2}{12x},$$

$$\iiint \Delta\, dy\, dx'\, dy' = \frac{p'q'}{4}\left(q'xy - p'\frac{y^2}{2}\right) + \frac{p'^3y^3}{36x} = \frac{p'q'}{4}\left(\frac{q'^2x^2}{p'} - \frac{q'^2x^2}{2p'}\right) + \frac{q'^3x^2}{36}$$

$$= q'^3x^2\left(\tfrac{1}{8}+\tfrac{1}{36}\right) = \tfrac{11}{72}q'^3x^2,$$

$$\iiiint \Delta\, dx\, dy\, dx'\, dy' = \tfrac{11}{216}p^3q'^3;$$

$$(2) \iint \Delta\, dx'\, dy' = -\frac{(q'x-yx')^3}{12xy} = \frac{q'^3x^3}{12xy} = \frac{q'^3x^2}{12y},$$

$$\iiint \Delta\, dy\, dx'\, dy' = \frac{q'^3x^2}{12}\log y = \frac{q'^3x^2}{12}\log\frac{p'q}{q'x},$$

$$\iiiint \Delta\, dx\, dy\, dx'\, dy' = \frac{q'^3x^3}{36}\log\frac{p'q}{q'x} + \frac{q'^3x^3}{108} = p^3q'^3\left(\tfrac{1}{108} + \tfrac{1}{36}\log\frac{p'q}{pq'}\right);$$

therefore $\qquad (1)+(2) = p^3q'^3\left(\tfrac{13}{216} + \tfrac{1}{36}\log\dfrac{p'q}{pq'}\right).$

2. When R is below PQ, $\Delta = \frac{1}{2}(yx'-xy')$, and $y' < y\dfrac{x'}{x}$. The several limits are as follows :—

$$\begin{array}{lll} \dfrac{y}{x} > \dfrac{q'}{p'} \left\{\begin{array}{l} y\dfrac{x'}{x} > q' \\ y\dfrac{x'}{x} < q' \end{array}\right. & \left\{\begin{array}{l} y'(o\ldots q'), \\ y'\left(o\ldots y\dfrac{x'}{x}\right), \end{array}\right. & \begin{array}{l} x'\left(q'\dfrac{x}{y}\ldots p'\right),\ y\left(\dfrac{q'}{p'}x\ldots q\right)\ldots.(1), \\ x'\left(o\ldots q'\dfrac{x}{y}\right),\qquad\quad ,,\qquad \ldots.(2), \end{array} \\ \dfrac{y}{x} < \dfrac{q'}{p'} \qquad ,, & \qquad ,, & x'(o\ldots p'),\quad y\left(o\ldots\dfrac{q'}{p'}x\right)\ldots.(3), \end{array}$$

therefore (1) $\int \Delta\, dy' = -\dfrac{(yx'-xy')^2}{4x} = \dfrac{y^2x'^2-(yx'-q'x)^2}{4x} = \dfrac{q'x'y}{2} - \dfrac{q'^2x}{4},$

$$\iint \Delta\, dx'\, dy' = \frac{q'x'^2y}{4} - \frac{q'^2xx'}{4} = \frac{p'^2q'}{4}y - \frac{p'q'^2}{4}x,$$

$$\iiint \Delta\, dy\, dx'\, dy' = \frac{p'^2q'}{8}y^2 - \frac{p'q'^2}{4}xy = \frac{p'^2q'q^2}{8} - \frac{p'q'^2qx}{4} + \frac{q'^3x^2}{8},$$

$$\iiiint \Delta\, dx\, dy\, dx'\, dy' = \frac{p'^2q'pq^2}{8} - \frac{p'q'^2p^2q}{8} + \frac{p^3q'^3}{24};$$

$$(2) \int \Delta\, dy' = \frac{y^2x'^2}{4x},$$

$$\iint \Delta\, dx'\, dy' = \frac{y^2x'^3}{12x} = \frac{y^2}{12x}\cdot\frac{q'^3x^3}{y^3} = \frac{q'^3x^2}{12y},$$

$$\iiint \Delta dy\, dx'\, dy' = \frac{q'^3x^2}{12}\log y = \frac{q'^3x^2}{12}\log\frac{p'q}{q'x},$$

$$\iiiint \Delta\, dx\, dy\, dx'\, dy' = p^3q'^3\left(\tfrac{1}{108} + \tfrac{1}{36}\log\frac{p'q}{pq'}\right);$$

$$(3)\ \iint \Delta\, dx'\, dy' = \frac{y^2x'^3}{12x} = \frac{p'^3y^2}{12x},$$

$$\iiint \Delta\, dy\, dx'\, dy' = \frac{p'^3y^3}{36x} = \frac{p'^3}{36x}\cdot\frac{q'^3x^3}{p'^3} = \frac{q'^3x^2}{36},$$

$$\iiiint \Delta\, dx\, dy\, dx'\, dy' = \frac{p^3q'^3}{108};$$

therefore $(1)+(2)+(3) = p^3q'^3\left(\tfrac{13}{216} + \tfrac{1}{36}\log\frac{p'q}{pq'}\right) + \frac{pqp'q'}{8}(p'q-pq')$.

But $\qquad p'q-pq' = p'(1-q')-pq' = p'-q' = q-p;$

therefore $(1)+(2)+(3) = p^3q'^3\left(\tfrac{13}{216} + \tfrac{1}{36}\log\frac{p'q}{pq'}\right) + \frac{pqp'q'}{8}(q-p)$.

This added to the former value gives

$$(\mathrm{AC}) = p^3q'^3\left(\tfrac{13}{108} + \tfrac{1}{18}\log\frac{p'q}{pq'}\right) + \frac{pqp'q'}{8}(q-p).$$

Similarly, interchanging p and p',

$$(\mathrm{BD}) = p'^3q'^3\left(\tfrac{13}{108} + \tfrac{1}{18}\log\frac{pq}{p'q'}\right) + \frac{pqp'q'}{8}(q-p').$$

Also, from what precedes,

$$(\mathrm{AA}+\mathrm{BB}+\mathrm{AB}) = p'^3\left(\tfrac{13}{216} - \frac{qq'}{18}\right),\quad (\mathrm{AD}) = q'^3\frac{pp'}{8},$$

$$(\mathrm{CD}+\mathrm{CC}+\mathrm{DD}) = p^3\left(\tfrac{13}{216} - \frac{qq'}{18}\right),\quad (\mathrm{BC}) = q^3\frac{pp'}{8}.$$

These several values will be reproduced when the positions of Q and R are interchanged, and therefore to include all positions they must be doubled. And the collective number of positions of Q and R being each represented by the area of the square, the total number of positions is unity. The sum of the areas is therefore in the present case equal to the average area. That is, collecting the last six values and doubling the result,

$$(\Delta) = \tfrac{13}{108} - \frac{pp'}{9} - \frac{qq'}{9} - \frac{5pp'qq'}{12} + \tfrac{13}{54}q'^3(1-3pp') + \frac{pp'qq'}{4}(q-q')$$
$$+ \frac{q'^3}{9}\left(p^3\log\frac{p'q}{pq'} + p'^3\log\frac{pq}{p'q'}\right),$$

which expresses the average area required in parts of the total area of the rectangle.

The transcendentals vanish only when P is in a side, in a diagonal, or at the centre.

With P in a side, $\quad (\Delta) = \tfrac{13}{108} - \frac{pp'}{9}$ (average $\tfrac{11}{108}$),
" at a corner, $\quad (\Delta) = \tfrac{13}{108}$,
" middle of side, $\quad (\Delta) = \tfrac{10}{108}$,
" at the centre, $\quad (\Delta) = \tfrac{5}{108}$.

In reference to the rectangle, p, q, p', q' are the four perpendiculars divided by the sides to which they are respectively parallel. Of these four fractions, q' is the least in value, and refers to the perpendicular which does not intersect a diagonal; q is the greatest, and involves the intersection of both diagonals. The intermediate values p, p' each alike intersect one diagonal, and with respect to them the formula is symmetrical.

Addendum.—As a check on the calculation, we may suppose P to vary its position on the rectangle, and by integrating the value of (Δ) with respect to p', q', so as to obtain the average area when all three points occupy all positions. To integrate for the right hand half of the upper diagonal space the limits are $q'\,(o \ldots p')$ and $p'\,(o \ldots \tfrac{1}{2})$.

Taking the first six terms of the expression, we have

$$u = \tfrac{13}{108} - \frac{pp'}{9} - \frac{qq'}{9} - \frac{5pp'qq'}{12} + \tfrac{13}{54}q'^3(1-3pp') + \frac{pp'qq'}{4}(q-q'),$$

$$\int u\,dq'\,(o \ldots p') = \tfrac{13}{108}p' - \frac{pp'^2}{9} - \frac{1}{9}\left(\frac{p'^2}{2} - \frac{p'^3}{3}\right) - \frac{5pp'}{12}\left(\frac{p'^2}{2} - \frac{p'^3}{3}\right)$$
$$+ \tfrac{13}{216}p'^4(1-3pp') + \frac{pp'}{4}\left(\frac{p'^2}{2} - p'^3 + \frac{p'^4}{2}\right)$$
$$= \tfrac{13}{108}p' - \tfrac{1}{6}p'^2 + \tfrac{7}{108}p'^3 + \tfrac{7}{216}p'^4 + \tfrac{1}{18}p'^5 + \tfrac{1}{18}p'^6.$$

As the point P traverses only $\tfrac{1}{8}$ of the total area, multiply by 8, and we get

$$8\iint u\,dp'dq'\,(o \ldots \tfrac{1}{2}) = 8\left(\tfrac{13}{216}\cdot\tfrac{1}{4} - \tfrac{1}{18}\cdot\tfrac{1}{8} + \tfrac{7}{432}\cdot\tfrac{1}{16} + \tfrac{7}{1080}\cdot\tfrac{1}{32}\right.$$
$$\left. + \tfrac{1}{108}\cdot\tfrac{1}{64} + \tfrac{1}{126}\cdot\tfrac{1}{128}\right) = \tfrac{8}{105}.$$

For the two remaining terms

$$u_1 = \frac{q'^3}{9}p^3 \log\frac{p'q}{pq'},$$

$$\int u_1 dq' = \frac{q'^4}{36}p^3 \log\frac{p'q}{pq'} + \frac{p^3}{36}\int q'^4\left(\frac{dq'}{1-q'} + \frac{dq'}{q'}\right)$$
$$= \frac{q'^4}{36}p^3 \log\frac{p'q}{pq'} + \frac{p^3}{36}\int\frac{q'^3dq'}{1-q'}$$

$$\text{(between limits)} = \frac{(1-p')^3}{36}\int\frac{p'^3dp'}{1-p'};$$

therefore $8\iint u_1 dp'dq' = -\dfrac{(1-p')^4}{18}\displaystyle\int\frac{p'^3dp'}{1-p'} + \tfrac{1}{18}\int p'^3(1-p')^3\,dp'$

$$= -\frac{(1-p')^4}{18}\left(\log\frac{1}{1-p'} - p' - \frac{p'^2}{2} - \frac{p'^3}{3}\right) + \tfrac{1}{18}\left(\frac{p'^4}{4} - \tfrac{3}{5}p'^5 + \tfrac{1}{2}p'^6 - \frac{p'^7}{7}\right)$$
$$= \frac{1}{18\,.\,16}\left\{-\log 2 + \tfrac{1}{2} + \tfrac{1}{8} + \tfrac{1}{24} + \left(\tfrac{1}{4} - \tfrac{3}{10} + \tfrac{1}{8} - \tfrac{1}{56}\right)\right\} = \tfrac{1}{288}\left(-\log 2 + \tfrac{76}{105}\right)$$
$$\ldots\ldots\ldots\ldots(\alpha).$$

$$u_2 = \frac{q'^3}{9}p'^3 \log\frac{pq}{p'q'},$$

$$\int u_2 dq' = \frac{q'^4}{36}p'^3 \log\frac{pq}{p'q'} + \frac{p'^3}{36}\int\frac{q'^3dq'}{1-q'}$$

$$\text{(limits)} = \frac{p'^7}{18}\log\frac{p}{p'} + \frac{p'^3}{36}\int\frac{p'^3dp'}{1-p'},$$

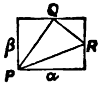
$$8\iint u_2 dp'dq' = \frac{p'^8}{18}\log\frac{p}{p'} + \tfrac{1}{18}\int\frac{p'^7dp'}{1-p'} + \frac{p'^4}{18}\int\frac{p'^3dp'}{1-p'} - \tfrac{1}{18}\int\frac{p'^7dp'}{1-p'}$$

$$=\frac{p'^3}{18}\log\frac{p}{p'}+\frac{p'^4}{18}\left(\log\frac{1}{1-p'}-p'-\frac{p'^2}{2}-\frac{p'^3}{3}\right)$$

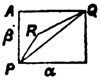
(limits) $=\frac{1}{18.16}(\log 2-\frac{1}{2}-\frac{1}{8}-\frac{1}{24})=\frac{1}{288}(\log 2-\frac{2}{3})$ ....(β).

therefore $\quad(\alpha)+(\beta)=\frac{1}{288}(\frac{76}{105}-\frac{2}{3})=\frac{1}{5040}$.

The value of (Δ) for the whole surface is therefore $\frac{8}{105}+\frac{1}{5040}=\frac{11}{144}$, which agrees with the result of other processes of a more direct and simple kind. Perhaps the following is as easy as any.



Let the triangle PQR be conceived to be circumscribed by the least rectangle having its sides (α, β) parallel to the sides of the square. Then the configurations of the points with reference to this rectangle will present two cases.



CASE 1.—One point P at a corner and the others in the remote sides. The average triangle has the latter points in the bisections of the sides, and its area is $\frac{3}{8}\alpha\beta$. The positions of P = 4, one for each corner; positions of Q, R $=\alpha\beta$; and permutations of P, Q, R = 6, giving in all $24\alpha\beta$ positions; and therefore areas $=\frac{3}{8}\alpha\beta\times 24\alpha\beta=9\alpha^2\beta^2$.

CASE 2.—Two points at opposite corners and the third anywhere on the surface. Here the average triangle evidently has R at the centre of gravity of APQ, and its area is $\frac{1}{6}\alpha\beta$. The positions of P, Q = 2; of R $=\alpha\beta$; and permutations = 6, giving $12\alpha\beta$ positions, and areas $=\frac{1}{6}\alpha\beta\times 12\alpha\beta=2\alpha^2\beta^2$.

Therefore in respect of the rectangle ($\alpha\beta$) the sum of the areas of the triangles $=9\alpha^2\beta^2+2\alpha^2\beta^2=11\alpha^2\beta^2$.

Now the number of positions of the rectangle ($\alpha\beta$) upon the surface of the given square, the side of which is unity, is evidently $(1-\alpha)(1-\beta)$. Therefore when P, Q, R range over the whole surface, we obtain

$$(\Delta)=\iint 11\alpha^2\beta^2 d\alpha d\beta\,(1-\alpha)(1-\beta)=\tfrac{11}{144}.$$

The same result is also readily found from the formula stated in Question 2471 (*Reprint*, Vol. VIII., p. 100) to the solution of which it is appended amongst other examples.

2484. (Proposed by J. GRIFFITHS, M.A.)—If A, B, C denote the angles of a plane triangle, L, M, N known quantities, and $f(x)\equiv x+x^{-1}$, find the value of k obtained by eliminating x and y from the equations

$$f(x)+2\cos A=k\mathrm{L},\quad f(y)+2\cos B=k\mathrm{M},\quad f(xy)+2\cos C=k\mathrm{N}.$$

Solution by W. H. LAVERTY; R. TUCKER, M.A.; W. CHADWICK; *and many others.*

Let $k\mathrm{L}-2\cos A=x+\dfrac{1}{x}=2\cos\theta$, and $k\mathrm{M}-2\cos B=y+\dfrac{1}{y}=2\cos\phi$,

then $\quad k\mathrm{N}-2\cos C=xy+\dfrac{1}{xy}=2\cos(\theta+\phi)$;

therefore $$\left\{(k\mathrm{N}-2\cos\mathrm{C})-2\cos\theta\cos\phi\right\}^2=4\sin^2\theta\sin^2\phi,$$

therefore $$(k\mathrm{N}-2\cos\mathrm{C})^2-(k\mathrm{L}-2\cos\mathrm{A})(k\mathrm{M}-2\cos\mathrm{B})(k\mathrm{N}-2\cos\mathrm{C})$$
$$=4-(k\mathrm{L}-2\cos\mathrm{A})^2-(k\mathrm{M}-2\cos\mathrm{B})^2,$$

which gives a cubic in k, the term not involving k being

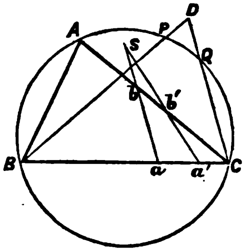
$$8\cos\mathrm{A}\cos\mathrm{B}\cos\mathrm{C}-4+4(\cos^2\mathrm{A}+\cos^2\mathrm{B}+\cos^2\mathrm{C}),$$

which is equal to zero, since $\mathrm{A}+\mathrm{B}+\mathrm{C}=180°$. Hence one value of k is $k=0$, and the other two values may be found from the quadratic equation

$$k^2\,\mathrm{LMN}-k(\Sigma\mathrm{L}^2+2\Sigma\mathrm{MN}\cos\mathrm{A})+4\Sigma\mathrm{L}\sin\mathrm{B}\sin\mathrm{C}=0.$$

2480. (Proposed by R. TUCKER, M.A.)—A circle is drawn round a triangle ABC; and, from any point D, lines DB, DC are drawn cutting the circle in P and Q, whose feet perpendicular lines with reference to ABC intersect in S; prove that the angle S is equal to the difference between the angles A and D.

Solution by T. DOBSON, B.A.; W. CHADWICK; *and others.*



Draw Pa, Qa', and Pb, Qb', perpendicular to BC, CA respectively; and produce ab, $a'b'$ to meet in S. Then the points abPC lie in a circle, as do also the points $a'b'$QC. Hence $\angle\,\mathrm{S}=ba\mathrm{B}-b'a'\mathrm{B}=b\mathrm{PC}-b'\mathrm{QC}=\mathrm{PCQ}=\mathrm{BPC}-\mathrm{D}=\mathrm{A}-\mathrm{D}$. If D be within the circle, $\mathrm{S}=\mathrm{D}-\mathrm{A}$.

COR.—If D′, D″ be the angles subtended by AC and AB at the point D, and S′, S″ angles analogous to S; we have
$$\mathrm{D}+\mathrm{D}'+\mathrm{D}''\pm(\mathrm{S}+\mathrm{S}'+\mathrm{S}'')=\pi.$$

2449 (From the TRINITY COLLEGE EXAMINATION PAPERS for 1865.)—If $\dfrac{f(x)}{\mathrm{F}(x)}$, where $f(x)$, $\mathrm{F}(x)$ are quadratic expressions in x, be expanded in ascending powers of x, prove that no term can vanish unless $\log\dfrac{f(\alpha)}{f(\beta)}+\log\left(\dfrac{\alpha}{\beta}\right)$ be a positive integer, α and β being the roots of $\mathrm{F}(x)$, supposed real and unequal.

Solution by J. McDowell, M.A., F.R.A.S.

Decomposing into partial fractions, we have

$$\frac{f(x)}{\mathrm{F}(x)} = \frac{f(x)}{(x-\alpha)(x-\beta)} = \frac{1}{\alpha-\beta}\left\{\frac{f(\alpha)}{x-\alpha} - \frac{f(\beta)}{x-\beta}\right\}$$

$$= \frac{f(\alpha)}{\alpha(\beta-\alpha)}\left\{1 + \frac{x}{\alpha} + \frac{x^2}{\alpha^2} + \ldots\ldots + \frac{x^n}{\alpha^n} + \ldots\ldots\right\}$$

$$+ \frac{f(\beta)}{\beta(\alpha-\beta)}\left\{1 + \frac{x}{\beta} + \frac{x^2}{\beta^2} \ldots\ldots + \frac{x^n}{\beta^n} + \ldots\ldots\right\}.$$

In order that the coefficient of x^n may vanish, we must have

$$\frac{f(\alpha)}{\alpha^{n+1}(\beta-\alpha)} + \frac{f(\beta)}{\beta^{n+1}(\alpha-\beta)} = 0, \text{ or } \frac{f(\alpha)}{f(\beta)} = \frac{\alpha^{n+1}}{\beta^{n+1}}$$

therefore
$$\log\frac{f(\alpha)}{f(\beta)} + \log\frac{\alpha}{\beta} = n+1.$$

2244. (Proposed by the late W. Lea.)—Form eleven symbols into sets, five symbols in each set, so that every combination of four symbols shall appear once in the sets.

Solution by the Proposer.

Let the symbols be $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}$, and put $n=11$; $p=5$; then the total combinations of four $= \frac{n(n-1)(n-2)(n-3)}{2.3.4}$; and the number contained in each set $= \frac{p(p-1)(p-2)(p-3)}{2.3.4}$;

therefore the number of sets $= \frac{n(n-1)(n-2)(n-3)}{p(p-1)(p-2)(p-3)} = 66.$

Hence we find that

Each symbol appears $\frac{(n-1)(n-2)(n-3)}{(p-1)(p-2)(p-3)} = 30$ times,

Each pair appears $\frac{(n-2)(n-3)}{(p-2)(p-3)} = 12$ times,

Each triad appears $\frac{n-3}{p-3} = 4$ times.

Proceeding tentatively with the formation of the several sets or combinations of five symbols so as to exhaust in succession the leading symbols a_1, a_2, a_3, a_4, a_5, we get the corresponding numbers of combinations

$$30+18+10+5+3 = 66,$$

the particulars of which are stated in the following table:

30 Combinations, Leading Symbol a_1.

1	$a_1\ a_2\ a_3$	$a_4\ a_5$
2		$a_6\ a_7$
3		$a_8\ a_9$
4		$a_{10}\ a_{11}$
5	$a_1\ a_2\ a_4$	$a_6\ a_9$
6		$a_7\ a_{11}$
7		$a_8\ a_{10}$
8	$a_1\ a_2\ a_5$	$a_6\ a_{10}$
9		$a_7\ a_8$
10		$a_9\ a_{11}$
11	$a_1\ a_2\ a_6$	$a_8\ a_{11}$
12	$a_1\ a_2\ a_7$	$a_9\ a_{10}$
13	$a_1\ a_3\ a_4$	$a_6\ a_8$
14		$a_7\ a_{10}$
15		$a_9\ a_{11}$
16	$a_1\ a_3\ a_5$	$a_6\ a_{11}$
17		$a_7\ a_9$
18		$a_8\ a_{10}$
19	$a_1\ a_3\ a_6$	$a_9\ a_{10}$
20	$a_1\ a_3\ a_7$	$a_8\ a_{11}$
21	$a_1\ a_4\ a_5$	$a_6\ a_7$
22		$a_8\ a_{11}$
23		$a_9\ a_{10}$
24	$a_1\ a_4\ a_6$	$a_{10}\ a_{11}$
25	$a_1\ a_4\ a_7$	$a_8\ a_9$
26	$a_1\ a_5\ a_6$	$a_8\ a_9$
27	$a_1\ a_5\ a_7$	$a_{10}\ a_{11}$
28	$a_1\ a_6\ a_7$	$a_8\ a_{10}$
29		$a_9\ a_{11}$
30	$a_1\ a_8\ a_9$	$a_{10}\ a_{11}$

18 Combinations, Leading Symbol a_2.

31	$a_2\ a_3\ a_4$	$a_6\ a_{11}$
32		$a_7\ a_8$
33		$a_9\ a_{10}$
34	$a_2\ a_3\ a_5$	$a_6\ a_9$
35		$a_7\ a_{10}$
36		$a_8\ a_{11}$
37	$a_2\ a_3\ a_6$	$a_8\ a_{10}$
38	$a_2\ a_3\ a_7$	$a_9\ a_{11}$
39	$a_2\ a_4\ a_5$	$a_6\ a_8$
40		$a_7\ a_9$
41		$a_{10}\ a_{11}$
42	$a_2\ a_4\ a_6$	$a_7\ a_{10}$
43	$a_2\ a_4\ a_8$	$a_9\ a_{11}$
44	$a_2\ a_5\ a_6$	$a_7\ a_{11}$
45	$a_2\ a_5\ a_8$	$a_9\ a_{10}$
46	$a_2\ a_6\ a_7$	$a_8\ a_9$
47	$a_2\ a_6\ a_9$	$a_{10}\ a_{11}$
48	$a_2\ a_7\ a_8$	$a_{10}\ a_{11}$

10 Combinations, Leading Symbol a_3.

49	$a_3\ a_4\ a_5$	$a_6\ a_{10}$
50		$a_7\ a_{11}$
51		$a_8\ a_9$
52	$a_3\ a_4\ a_6$	$a_7\ a_9$
53	$a_3\ a_4\ a_8$	$a_{10}\ a_{11}$
54	$a_3\ a_5\ a_6$	$a_7\ a_8$
55	$a_3\ a_5\ a_9$	$a_{10}\ a_{11}$
56	$a_3\ a_6\ a_7$	$a_{10}\ a_{11}$
57	$a_3\ a_6\ a_8$	$a_9\ a_{11}$
58	$a_3\ a_7\ a_8$	$a_9\ a_{10}$

5 Combinations, Leading Symbol a_4.

59	$a_4\ a_5\ a_6$	$a_9\ a_{11}$
60	$a_4\ a_5\ a_7$	$a_8\ a_{10}$
61	$a_4\ a_6\ a_7$	$a_8\ a_{11}$
62	$a_4\ a_6\ a_8$	$a_9\ a_{10}$
63	$a_4\ a_7\ a_9$	$a_{10}\ a_{11}$

3 Combinations, Leading Symbol a_5.

64	$a_5\ a_6\ a_7$	$a_9\ a_{10}$
65	$a_5\ a_6\ a_8$	$a_{10}\ a_{11}$
66	$a_5\ a_7\ a_8$	$a_9\ a_{11}$

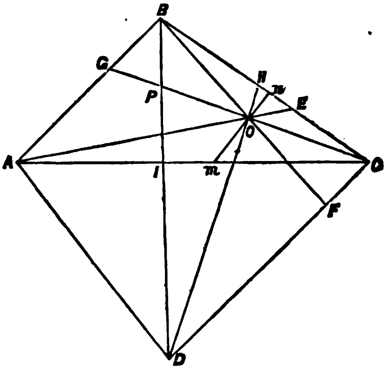
[NOTE.—After forming the obvious arrangements Nos. 1, 2, 3, 4, put down for subsidiary use the pairs

6	7	8	9
9	11	10	11
10	8	11	10

Then, from an inspection of these, all the arrangements from 5 to 20, both inclusive, are readily found.]

2541. (Proposed by T. SAVAGE, M.A.)—If the diagonals of a convex quadrilateral divide each other in the ratios $\lambda : 1-\lambda$ and $\mu : 1-\mu$, the chance that the line passing through two points taken at random within the quadrilateral will meet both the diagonals at points within the quadrilateral is $\frac{1}{3}\left\{1+2(\lambda+\mu)-2(\lambda^2+\mu^2)\right\}$. Thus, for a parallelogram the chance is $\frac{2}{3}$.

I. *Solution by* STEPHEN WATSON.



Let ABCD be the quadrilateral; I the intersection of the diagonals; and through O, one of the points, draw AE, BF, CG, DH, as in the diagram, and mOn parallel to AB. Then in order that the line through O, P may cut only *one diagonal*, P must lie in one of the triangles AOG, BOH, COE, DOF. Put AC $= a$, BD $= b$, BI $= h$, and AIB $= \alpha$; then when the points O are uniformly distributed over mn, the points G will be uniformly spread over AB, and the same holds good for every other line parallel to mn; hence while O takes every position in the triangle BAC the average distance of G from A is $\frac{1}{2}$AB, and of G from the line AC is $\frac{1}{2}h\sin\alpha$, and the same holds good of the distance of E from AC; also the average distance of O from AC is $\frac{1}{3}h\sin\alpha$. Hence the sum of the areas of the triangles AOG, COE, while O lies in ABC, is

$$a\left(\tfrac{1}{2}-\tfrac{1}{3}\right)h\sin\alpha \,.\, \Delta\,\mathrm{ABC} = \tfrac{1}{12}a^2h^2\sin^2\alpha,$$

and the chance of P lying in one of the same triangles is

$$\tfrac{1}{12}a^2h^2\sin^2\alpha \div \tfrac{1}{4}a^2b^2\sin^2\alpha = \frac{h^2}{3b^2} = \tfrac{1}{3}\mu^2;$$

also the like chance when O is in ADC is $= \frac{1}{3}(1-\mu)^2$. Similarly the chance of P lying in one of the triangles BOH, DOF is $\frac{1}{3}\lambda^2$ or $\frac{1}{3}(1-\lambda)^2$ according as O is to the left or right of BD; hence the chance of the line through O, P cutting *one diagonal* is

$$\tfrac{1}{3}\left\{\lambda^2+\mu^2+(1-\lambda)^2+(1-\mu)^2\right\}\ldots\ldots\ldots\ldots\ldots(1);$$

and therefore the chance of cutting *both diagonals* is

$$1-(1)=\tfrac{1}{3}\left\{1+2(\lambda+\mu)-2(\lambda^2+\mu^2)\right\}\ldots\ldots\ldots\ldots(2).$$

In the case of a parallelogram $\lambda=\mu=\frac{1}{2}$, and the chances (1) and (2) become $\frac{1}{3}$ and $\frac{2}{3}$ which are as 1 : 2.

When the line OP merely *joins* O, P. In this case, in order that OP may cut *both diagonals*, O and P must lie one in the triangle AIB and the other in the triangle CID, the chance of which is $2\lambda\mu(1-\lambda)(1-\mu)$, or one in the triangle BIC and the other in the triangle AID, the chance of which is the same; hence the chance of cutting *both diagonals* is

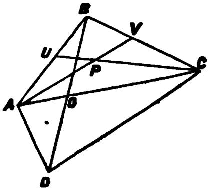
$$4\lambda\mu(1-\lambda)(1-\mu).$$

Similarly the chances of cutting *one diagonal* and *neither diagonal*, are

$$4\lambda\mu(1-\lambda)(1-\mu),\quad 4\left\{\lambda^2\mu^2+(1-\lambda)^2(1-\mu)^2\right\}.$$

In the case of a parallelogram, the three chances just found become $\frac{1}{4}$, $\frac{1}{4}$, $\frac{1}{2}$ which are as 1 : 1 : 2.

II. *Solution by* Capt. A. R. Clarke, R.E., F.R.S.



If through any point P in a triangle ABC, lines APV, CPU be drawn meeting the opposite sides in V and U, it may be easily shown by direct integration, or may be inferred from Professor Sylvester's Solution to Question 2371 (*Reprint*, Vol. VIII., p. 36), that (dA being an element of area at P)

$$\int (\mathrm{AUP}+\mathrm{CVP})\,dA = \tfrac{1}{3}(\mathrm{ABC})^2,$$

the integration extending over the area ABC. Hence the total number of pairs of points which determine a line cutting the segment OB of the diagonal BD *only*, is $\frac{1}{3}\cdot\frac{1}{4}\mathrm{OB}^2\cdot\mathrm{AC}^2\cdot\sin^2\mathrm{O}$. Therefore the number of pairs of points determining lines which cut BD *only*, is $\frac{1}{12}(\mathrm{OB}^2+\mathrm{DO}^2)\,\mathrm{AC}^2\sin^2\mathrm{O}$. Consequently the number of pairs of points determining lines which cut only *one* diagonal (within the figure) is

$$\tfrac{1}{12}\left\{(\mathrm{OB}^2+\mathrm{OD}^2)\,\mathrm{AC}^2+(\mathrm{AO}^2+\mathrm{OC}^2)\,\mathrm{BD}^2\right\}\sin^2\mathrm{O}.$$

And the total number of pairs is $\frac{1}{4}\mathrm{AC}^2\cdot\mathrm{BD}^2\sin^2\mathrm{O}$. Hence the chance of cutting one diagonal only is

$$\frac{1}{3}\cdot\frac{\mathrm{OB}^2+\mathrm{OD}^2}{\mathrm{BD}^2}+\frac{1}{3}\cdot\frac{\mathrm{AO}^2+\mathrm{OC}^2}{\mathrm{AC}^2}=\tfrac{1}{3}\left\{\lambda^2+(1-\lambda)^2+\mu^2+(1-\mu)^2\right\}.$$

Subtracting this from unity, we get the stated result for the chance of cutting both diagonals.

2530. (Proposed by Professor Cayley.)—Trace the curve $\dfrac{1}{\sqrt{z}}+\dfrac{1}{\sqrt{(x+iy)}}+\dfrac{1}{\sqrt{(x-iy)}}=0$, where the coordinates x, y, z are the perpendicular distances of the current point P from the sides of an equilateral triangle, the coordinates being positive for a point within the triangle.

Solution by the Proposer.

The form of the equation shows that the curve is a tricuspidal quartic, having a real cusp at the point ($x=0$, $y=0$), and two imaginary cusps at the points $(z=0,\ x+iy=0)$ and $(z=0,\ x-iy=0)$. The rationalised form of the equation is

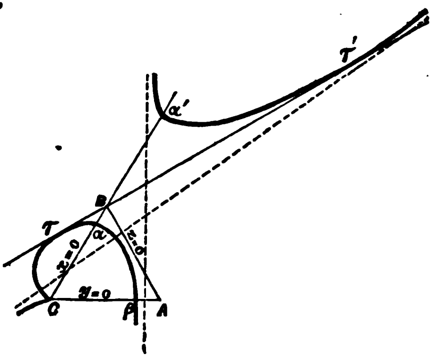
$$(x^2+y^2)^2-4zx\,(x^2+y^2)-4z^2y^2=0.$$

$x=0$ gives $y^2(y^2-4z^2)=0$, the point C twice, and two other real points α, α' on the line BC.

$y=0$ gives $x^3(x-4z)=0$, the point C three times, and a real point β on the line CA.

It is easy to find that there is a double tangent $z+2x=0$, viz. $z+2x=0$ gives $(3x^2-y^2)^2=0$, two points τ, τ' (each twice) on the line in question.

Laying down these points, it is easy to see that the curve must have two real asymptotes, and that the form is as shown in the figure.



2524. (Proposed by J. J. WALKER, M.A.)—Show that the equation to the circle circumscribing the triangle formed by the three straight lines $ax+by+c=0$, $a'x\ldots\ldots=0$, $a''x\ldots\ldots=0$ may be thrown into the form

$$(ab'c'')\,(x^2+y^2)+\Sigma\,\frac{(a)^2+(b)^2}{(c)}\,(ax+by+c)=0,$$

where $(ab'c'')$ is the determinant whose constituents are the coefficients in the three linear equations, and $(a)\ (b)\ (c)\ (a')\ldots\ldots$ its first minors.

Solution by the PROPOSER.

1. The condition that four points $(x, y), \ldots\ldots, (x_3, y_3)$ should be concyclic (being the analytic expression of Euc. III., 21) is, in rectangular Cartesian coordinates,

$$\frac{(x-x_2)\,(y-y_1)-(x-x_1)\,(y-y_2)}{(x-x_1)\,(x-x_2)+(y-y_1)\,(y-y_2)}-\frac{(x_3-x_2)\,(y_3-y_1)-(x_3-x_1)\,(y_3-y_2)}{(x_3-x_1)\,(x_3-x_2)+(y_3-y_1)\,(y_3-y_2)}=0,$$

or $\Sigma \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} (x^2+y^2)=0$, as is well known.

This may be thrown into the form

$$\begin{aligned}&(1x_2y_3)\,(x^2+y^2)\\ &+\left\{(x_1^2+y_1^2)\,(y_3-y_2)+(x_2^2+y_2^2)\,(y_1-y_3)+(x_3^2+y_3^2)\,(y_2-y_1)\right\}x\\ &+\left\{(x_1^2+y_1^2)\,(x_2-x_3)+(x_2^2+y_2^2)\,(x_3-x_1)+(x_3^2+y_3^2)\,(x_1-x_2)\right\}y\\ &+(x_1^2+y_1^2)\,(x_3y_2-x_2y_3)+(x_2^2+y_2^2)\,(x_1y_3-x_3y_1)+(x_3^2+y_3^2)\,(x_2y_1-x_1y_2)\\ &=0\ldots\ldots(1)\end{aligned}$$

Now if (x_1, y_1) is the intersection of the lines $a'x + \ldots = 0$, $a' x + \ldots = 0$, (x_2, y_2) of $ax + \ldots = 0$, $a''x \ldots = 0$, (x_3, y_3) of $ax + \ldots 0$, $a'x + \ldots = 0$;

then $$x_1 = \frac{(a)}{(c)},\ y_1 = \frac{(b)}{(c)},\ x_2 = \frac{(a')}{(c')},\ y_2 = \frac{(b')}{(c')}\ \ x_3 = \frac{(a'')}{(c'')},\ y_3 = \frac{(b'')}{(c'')},$$

whence $$(1x_2y_3) = \frac{(ab'c'')^2}{(c)\,(c')\,(c'')},\ y_3 - y_2 = \frac{a\,(ab'c'')}{(c')\,(c'')},\ y_1 - y_3 = \frac{a'\,(ab'c'')}{(c)\,(c'')},$$

$$y_2 - y_1 = \frac{a''\,(ab'c'')}{(c)\,(c)'},\ x_2 - x_3 = \frac{b\,(ab'c'')}{(c')\,(c'')},\ x_3 - x_1 = \frac{b'\,(ab'c'')}{(c)\,(c'')},$$

$$x_1 - x_2 = \frac{b''\,(ab'c'')}{(c)\,(c')},\ x_3y_2 - x_2y_3 = \frac{c\,(ab'c'')}{(c)},\ x_1y_3 - x_3y_1 = \frac{c'(ab'c'')}{(c')},$$

$$x_2y_1 - x_1y_2 = \frac{c''\,(ab'c'')}{(c'')}.$$

Substituting these values in the equation (1) above, it becomes divisible by $\dfrac{(ab'c'')}{(c)\,(c')\,(c'')}$, which reduces it to the form given in the question.

2344. (Proposed by J. Griffiths, M.A.)—Let P be any point on a rectangular hyperbola whose centre is O; then if with centre P and radius PO a circle be described, an infinite number of triangles can be inscribed in the circle, and circumscribed to the hyperbola. Let ABC be any one of these triangles; l, m, n the points where the sides BC, CA, AB touch the hyperbola: it is required to prove that the straight lines Al, Bm, Cn intersect on the circle.

Solution by James Dale.

Taking any one of the triangles ABC as that of reference, the equation to the inscribed conic will be of the form

$$(lx)^{\frac{1}{2}} + (my)^{\frac{1}{2}} + (nz)^{\frac{1}{2}} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The lines joining A, B, C to the points of contact of the opposite sides intersect in the point $lx = my = nz$, which lies on the circumscribing circle if

$$l \sin A + m \sin B + n \sin C = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

But since (1) is a rectangular hyperbola,

$$l^2 + m^2 + n^2 + 2mn \cos A + 2nl \cos B + 2lm \cos C = 0 \ldots\ldots\ldots (3);$$

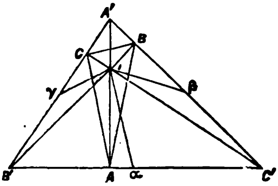
and since (1) passes through the centre of the circumscribing circle, we have

$$l^2 \cos^2 A + m^2 \cos^2 B + n^2 \cos^2 C - 2mn \cos B \cos C - \&c. = 0 \ldots\ldots (4).$$

From $\left\{(1) - (2)\right\}^{\frac{1}{2}}$ we obtain $l \sin A + m \sin B + n \sin C = 0$; therefore, by (2), A$l$, B$m$, C$n$ intersect on the circumscribed circle.

2404. (Proposed by J. Griffiths, M.A.)—Let A′, B′, C′ be the centre of the escribed circles of a triangle ABC; I that of the inscribed circle, α, β, γ the middle points of the sides of the triangle A′B′C′. On the segments IA′, IB′, IC′ take X, Y, Z such that IX $=\frac{1}{4}$IA′, IY $=\frac{1}{4}$IB′, IZ $=\frac{1}{4}$IC′, then the circle through X, Y, Z will bisect each of the six segments IA, IB, IC, Iα, Iβ, Iγ.

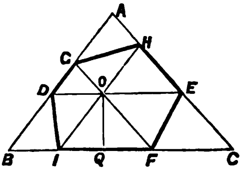
Solution by J. Dale; G. A. Ogilvie; *and others.*



The circle circumscribing ABC is the nine-point circle of A′B′C′, and I is the intersection of the perpendiculars of A′B′C′; therefore the middle points of IA′, IB′, IC′ lie on the circle ABC, and X, Y, Z will lie on a circle of half the dimensions of ABC, having its centre on the middle point of the line joining I to the centre of ABC, and which bisects every line drawn from I to the circumference of ABC (see *Note on the Nine-Point Circle, Reprint*, Vol. VII., p. 86); and which consequently passes through the middle points of IA, IB, IC, Iα, Iβ, Iγ.

2318. (Proposed by S. Watson.)—Through every point O within a triangle ABC, parallels DE, FG, HI are drawn to BC, CA, AB respectively; show that the average area of the hexagon DIFEHG is three-fourths of the triangle.

Solution by the Proposer.



Draw OQ perpendicular to BC; and put DO $= x$, OQ $= y$, and $p =$ the perpendicular from A on BC. Then by similar triangles

$p : a = p - y :$ DE, $\therefore$ DE $= \dfrac{a(p-y)}{p}$, and

$p : a = y :$ IF, $\therefore$ IF $= \dfrac{ay}{p}$; hence the area of the trapezoid IDOF is

$$\tfrac{1}{2}\text{OQ}(\text{IF}+\text{DO}) = \frac{y}{2p}(ay+px)\ldots\ldots\ldots\ldots\ldots(1).$$

Now an element at O is $dydx$, and triangle ABC $= \Delta =$ the total number of positions of O; therefore the average area of the trapezoid IDOF is

$$\frac{1}{\Delta}\int_0^p \int_0^{\frac{a(p-y)}{p}} \frac{y}{2p}(ay+px)\,dydx = \frac{\Delta}{4}.$$

Similarly $\frac{1}{4}\Delta$ is the average area of each of the trapezoids EFOH, GHOD; therefore the average area required is $\frac{3}{4}\Delta$.

Otherwise: the area of BDI is $\frac{1}{2}xy$; therefore the average area is

$$\frac{1}{\Delta}\int_0^p\int_0^{\frac{a(p-y)}{p}}\tfrac{1}{2}yx\,dy\,dx=\frac{\Delta}{12},$$

and this is also the average area of each of the triangles CFE, HGA; therefore $\Delta-\frac{1}{4}\Delta$, or $\frac{3}{4}\Delta$, is the average area required, as before.

2522. (Proposed by W. K. CLIFFORD, B.A.)—Prove (1) that the perpendiculars of a circular triangle have a common radical axis; and (2) that if the perpendiculars from the pairs of vertices of one circular triangle on the sides of another meet in a point, then *vice versâ*. (*Def.*—A, B, C being circles, a circle coaxal with A, B, and orthogonal to C, is called the perpendicular from AB on C.)

Solution by the REV. R. TOWNSEND, F.R.S.

1. If A, B, C be the centres of the three circles forming the triangle; a, b, c their three radii; α, β, γ their three angles of intersection two and two; and X, Y, Z the centres of the three perpendiculars; then since (see *Modern Geometry*, Vol. I., art. 193)

$$\frac{\mathrm{BX}}{\mathrm{CX}}=\frac{b}{c}\cdot\frac{\cos\gamma}{\cos\beta},\quad\frac{\mathrm{CY}}{\mathrm{AY}}=\frac{c}{a}\cdot\frac{\cos\alpha}{\cos\gamma},\quad\frac{\mathrm{AZ}}{\mathrm{BZ}}=\frac{a}{b}\cdot\frac{\cos\beta}{\cos\alpha},$$

therefore at once, by composition of ratios, $\dfrac{\mathrm{BX}}{\mathrm{CX}}\cdot\dfrac{\mathrm{CY}}{\mathrm{AY}}\cdot\dfrac{\mathrm{AZ}}{\mathrm{BZ}}=1,$

and therefore, &c. [See *Modern Geometry*, Vol. I., art. 190.]

2. If A, B, C be the centres, and a, b, c the radii of the three circles forming one triangle; A′, B′, C′ and a', b', c' those of the three forming the other triangle; X, Y, Z and X′, Y′, Z′ the two triads of centres of the two triads of perpendiculars, and $\widehat{AB'}$ and $\widehat{AC'}$, $\widehat{BC'}$ and $\widehat{BA'}$, $\widehat{CA'}$ and $\widehat{CB'}$ the six angles of intersection of the three pairs of non-corresponding sides of the two triangles; then since (*Mod. Geom.*, Vol. I., art. 193)

$$\frac{\mathrm{BX}}{\mathrm{CX}}=\frac{b}{c}\cdot\frac{\cos\widehat{\mathrm{BA'}}}{\cos\widehat{\mathrm{CA'}}},\quad\frac{\mathrm{CY}}{\mathrm{AY}}=\frac{c}{a}\cdot\frac{\cos\widehat{\mathrm{CB'}}}{\cos\widehat{\mathrm{AB'}}},\quad\frac{\mathrm{AZ}}{\mathrm{BZ}}=\frac{a}{b}\cdot\frac{\cos\widehat{\mathrm{AC'}}}{\cos\widehat{\mathrm{BC'}}},$$

$$\frac{\mathrm{B'X'}}{\mathrm{C'X'}}=\frac{b'}{c'}\cdot\frac{\cos\widehat{\mathrm{B'A}}}{\cos\widehat{\mathrm{C'A}}},\quad\frac{\mathrm{C'Y'}}{\mathrm{A'Y'}}=\frac{c'}{a'}\cdot\frac{\cos\widehat{\mathrm{C'B}}}{\cos\widehat{\mathrm{A'B}}},\quad\frac{\mathrm{A'Z'}}{\mathrm{B'Z'}}=\frac{a'}{b'}\cdot\frac{\cos\widehat{\mathrm{A'C}}}{\cos\widehat{\mathrm{B'C}}},$$

and since consequently, by composition of ratios,

$$\frac{\mathrm{BX}}{\mathrm{CX}}\cdot\frac{\mathrm{CY}}{\mathrm{AY}}\cdot\frac{\mathrm{AZ}}{\mathrm{BZ}}=\frac{\mathrm{B'X'}}{\mathrm{C'X'}}\cdot\frac{\mathrm{C'Y'}}{\mathrm{A'Y'}}\cdot\frac{\mathrm{A'Z'}}{\mathrm{B'Z'}},$$

therefore if either product $= 1$, so is the other also, and therefore &c. [*Modern Geometry*, Vol. I., art. 190.)

By inversion from any arbitrary point in space the two preceding properties are at once seen to be true, the former for any arbitrary triad, and the latter for any two arbitrary triads, of circles on the surface of a sphere.

2387. (Proposed by S. Roberts, M.A.)—Given a point O and a straight line A, another straight line B passes through O and meets A in a point whose mth polar with regard to a curve of the nth order meets B in $n-m$ points. Required the locus of the points when B revolves about O. Show, by means of the general expression, the locus of the middle points of chords of a conic which pass through a given point.

I. *Solution by* James Dale.

Let (f, g, h) be the given point; (l, m, n) the given line; (x, y, z) a point on the required locus.

The mth polar of any point (x', y', z') on the given line, with regard to a curve of the nth order, is

$$\left(x'\frac{d}{dx}+y'\frac{d}{dy}+z'\frac{d}{dz}\right)^m\phi=0,$$

the differential symbols operating on ϕ only; and as ϕ is of the order n, this will be of the $(n-m)$th order. Since (x', y', z') lies on the given line, we have the condition $lx'+my'+nz'=0$; and since (f, g, h), (x', y', z'), (x, y, z) lie on the same straight line, we have the condition

$$\begin{vmatrix} y & z \\ g & h \end{vmatrix} x' + \begin{vmatrix} z & x \\ h & f \end{vmatrix} y' + \begin{vmatrix} x & y \\ f & g \end{vmatrix} z' = 0,$$

$$\therefore \frac{x'}{(lf+mg+nh)\,x-f(lx+my+nz)} = \frac{y'}{(lf+mg+nh)\,y-g\,(lx+my+nz)} = \frac{z'}{(lf+mg+nh)\,z-h(lx+my+nz)};$$

and substituting these values in the equation of the polar, we find the equation to the locus of x, y, z to be

$$\left\{(lf+mg+nh)\left(x\frac{d}{dx}+y\frac{d}{dy}+z\frac{d}{dz}\right) - (lx+my+nz)\left(f\frac{d}{dx}+g\frac{d}{dy}+h\frac{d}{dz}\right)\right\}^m\phi=0.$$

When ϕ is a conic, this becomes

$$2\phi(x, y, z) = \frac{lx+my+nz}{lf+mg+nh}\left(x\frac{d}{df}+y\frac{d}{dg}+z\frac{d}{dh}\right)\phi,$$

which represents a conic passing through the given point, and the pole of the given line, and through the intersections of the polar of the given point, and the given line with the conic.

When the given line is at infinity the locus becomes

$$\phi(x, y, z) = \frac{ax+by+cz}{4\Delta}\left(x\frac{d}{df}+y\frac{d}{dg}+z\frac{d}{dh}\right)\phi,$$

which represents a conic similar and similarly situated to the given conic, passing through its centre and the given point. This conic is the locus of the middle points of all chords of ϕ which pass through (f, g, h).

II. *Solution by the* PROPOSER.

Using trilinear coordinates (x, y, z), let $z=0$ be the given line A, and let the coordinates of the point O be $(x=0, y=0)$. The line $xy_1-x_1y=0$...(1) passes through O and meets A in the point $(x_1, y_1, 0)$ the mth polar of which relative to the curve of the nth degree $u=0$ is $\left(x_1\frac{d}{dx}+y_1\frac{d}{dy}\right)^m u=0$ ...(2).

Eliminate x_1, y_1 between (1) and (2), and we get the required equation

$$\left\{x^m\frac{d^m}{dx^m}+mx^{m-1}\frac{d^m}{dx^{m-1}\,dy}+\ldots+y^m\frac{d^m}{dy^m}\right\}u=0 \,\ldots\ldots\ldots (3).$$

The function here represented is a mixed concomitant of the function u. If we adopt the general forms $\alpha x+\beta y+\gamma z=0$ for the line A and (x_1, y_1, z_1) for the point O, the resulting equation is

$$\left\{\Big((z_1x-x_1z)\gamma-(x_1y-xy_1)\beta\Big)\frac{d}{dx}+\Big((x_1y-xy_1)\alpha-(y_1z-z_1y)\beta\Big)\frac{d}{dy}\right.$$
$$\left.+\Big((y_1z-z_1y)\beta-(z_1x-x_1z)\alpha\Big)\frac{d}{dz}\right\}^m u=0,$$

the differential symbols operating on u only.

In the case of a conic the equation (3) becomes

$$x\frac{du}{dx}+y\frac{du}{dy}=0.$$

And if z is the line at infinity, this represents the locus of the middle points of chords through the origin. (Salmon's *Conics*, p. 95, Ex. 2, 3rd ed.)

2469. (Proposed by R. TUCKER, M.A.) — Transform the equation $x^4+px^3+qx^2+rx+s=0$, with roots $(\alpha, \beta, \gamma, \delta)$, to one whose roots are $\beta\gamma\delta+\beta\gamma+\gamma\delta+\delta\beta+\beta+\gamma+\delta$, &c.

Solution by the PROPOSER; W. CHADWICK; *and others.*

Since $\beta\gamma\delta+\beta\gamma+\gamma\delta+\delta\beta+\beta+\gamma+\delta=\frac{s}{x}+q+x(p+x)-p-x,$

we have to eliminate x between

$$xy=s+(q-p)x+x^2(p-1)+x^3 \text{ and } x^4+px^3+qx^2+rx+s=0;$$

whence we have, if $a = -p+q-r+s$,

$$(a-y)^4+p(1+y)(a-y)^3+q(1+y)^2(a-y)^2$$
$$+r(1+y)^3(a-y)+s(1+y)^4=0,$$

or
$$(a+1)y^4+(r+3p-2q)(a+1)y^3$$
$$+\left\{a^2(q-3p+6)+a(3p-4q+3r)+q-3r+6s\right\}y^2$$
$$+\left\{a^3(p-4)+a^2(2q-3p)+a(3r-2q)+4s-r\right\}y$$
$$+a^4+pa^3+qa^2+ra+s=0.$$

If -1 be a root of the given equation, then the transformed will have three roots equal to -1.

Example: $x^4-4x^3-7x^2+34x-24=0$, (roots 2, -3, 4, 1), transformed equation $x^4+36x^3-558x^2-30020x-245427=0$.

1977. (Proposed by M. W. CROFTON, B.A.)—If three lengths are taken at random, find the chance that it will be possible to form a triangle with them; the extreme limit, whatever its magnitude, being supposed the same for all three.

I. *Solution by the* PROPOSER.

Let AB be the given limit; then the question will be, taking three points 1, 2, 3 at random in AB, to find the chance that A1, A2, A3 may form a triangle.

As the result must be independent of the limit AB, we may take A3, the distance of the farthest point from A, as the limit; *i.e.*, suppose A3 a fixed distance, and 1, 2 taken at random in it. Put A1 $= x$, A2 $= y$, A3 $= a$; then we want the chance that $x+y>a$. Now to every pair of values (x, y) corresponds another pair $(a-x, a-y)$; this we see clearly by conceiving the distances measured from the end 3 of the line A3, instead of from A.

But if $x+y>a$, we have $a-x+a-y<a$.

Hence to every favourable case corresponds one unfavourable case; the chance required is therefore $\frac{1}{2}$.

II. *Solution by* STEPHEN WATSON.

Let x, y, z be the lengths, and a the limiting magnitude. Then, multiplying by 6, the number of ways in which x, y, z can be interchanged, we need only consider the question under the condition $x<y<z$, and thus the possibility of forming a triangle with x, y, z is $x+y>z$; therefore, when $x<\frac{1}{2}z$, the limits of y are $z-x$ and z; but when $x>\frac{1}{2}z$, the limits are x and z. Also the number of lengths taken $=a^3$; hence the required chance is

$$p=\frac{6}{a^3}\int_0^a dz\left\{\int_0^{\frac{1}{2}z}dx\int_{z-x}^{z}dy+\int_{\frac{1}{2}z}^{z}dx\int_x^z dy\right\}=\tfrac{1}{2};$$

therefore the chance is an *even* one whatever be the magnitude of a.

2504. (Proposed by R. TUCKER, M.A.)—O is the centre of the circumscribing circle of a triangle ABC; PK, PL, PM, drawn parallel to CO, BO, AO respectively, meet AB, AC, BC in K, L, M; if KLM be a straight line, then the locus of P will be the conic

$$\frac{a}{\alpha}\cos A\cos(B-C)+\frac{b}{\beta}\cos B\cos(C-A)+\frac{c}{\gamma}\cos C\cos(A-B)=0.$$

I. *Solution by the* Rev. J. L. KITCHIN, M.A.; *the* PROPOSER; *and others.*

The equations to the lines joining O with A, B, C in order, are

$$\beta\cos C-\gamma\cos B=0,\quad \gamma\cos A-\alpha\cos C=0,\quad \alpha\cos B-\beta\cos A=0\ \ldots.(1).$$

Three lines through a point $(\alpha_1, \beta_1, \gamma_1)$ parallel to (1) in order, are

$$\left.\begin{array}{l}(\beta\gamma_1-\gamma\beta_1)\cos(B-C)+(\alpha\gamma_1-\gamma\alpha_1)\cos B+(\beta\alpha_1-\alpha\beta_1)\cos C=0\\ (\gamma\alpha_1-\alpha\gamma_1)\cos(C-A)+(\beta\alpha_1-\alpha\beta_1)\cos C+(\gamma\beta_1-\beta\gamma_1)\cos A=0\\ (\alpha\beta_1-\beta\alpha_1)\cos(A-B)+(\gamma\beta_1-\beta\gamma_1)\cos A+(\alpha\gamma_1-\gamma\alpha_1)\cos B=0\end{array}\right\}\ldots.(2).$$

The condition that the intersections of (2) in order with $\alpha=0$, $\beta=0$, $\gamma=0$ may all lie on a straight line is easily found to be

$$\Big\{\alpha_1\cos(A-B)+\gamma_1\cos A\Big\}\Big\{\gamma_1\cos(C-A)+\beta_1\cos C\Big\}\Big\{\beta_1\cos(B-C)+\alpha_1\cos B\Big\}$$
$$+\Big\{\beta_1\cos(A-B)+\gamma_1\cos B\Big\}\Big\{\alpha_1\cos(C-A)+\beta_1\cos A\Big\}$$
$$\times\Big\{\gamma_1\cos(B-C)+\alpha_1\cos C\Big\}=0.$$

Multiplying these out, and performing some obvious reductions, we get

$$\Big\{\sin^2 A\cos A\cos(B-C)+\sin^2 B\cos B\cos(C-A)+\sin^2 C\cos C\cos(A-B)\Big\}\alpha_1\beta_1\gamma_1$$
$$+\beta_1\gamma_1^2\cos A\cos(B-C)\sin A\sin C+\&c.=0,$$

which, suppressing the accents, splits up into

$$(\alpha\sin A+\beta\sin B+\gamma\sin C)\Big\{\beta\gamma\sin A\cos A\cos(B-C)+\&c.\Big\}=0.$$

Now, obviously, the first factor put equal to 0 cannot be locus of P; hence the equation of the locus of P is $\beta\gamma\sin A\cos A\cos(B-C)+\&c.=0$, which is the same as that given in the question.

II. *Solution by* STEPHEN WATSON; W. CHADWICK; *and others.*

For brevity let l, m, n stand for cos A, cos B, cos C respectively, then the equations of AO, BO, CO are contained in

$$\frac{\alpha}{l}=\frac{\beta}{m}=\frac{\gamma}{n}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and those of lines parallel to the same may be written

$$n\beta-m\gamma+p\,(a\alpha+b\beta+c\gamma)=0\ \ldots\ldots\ldots\ldots\ldots\ldots(2),$$
$$l\gamma-n\alpha+p_1(a\alpha+b\beta+c\gamma)=0\ \ldots\ldots\ldots\ldots\ldots\ldots(3).$$
$$m\alpha-l\beta+p_2(a\alpha+b\beta+c\gamma)=0\ \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

The equation of a line through the intersections of (2) with AC and (3) with AB, is

$$\alpha - \frac{l-bp_2}{m+ap_2}\beta - \frac{l+cp_1}{n-ap_1}\gamma = 0 \;\dots\dots\dots\dots\dots\dots (5);$$

hence the condition that (2) and (5) shall meet BC in the same point is

$$\frac{m-cp}{n+bp} = -\frac{(l+cp_1)(m+ap_2)}{(n-ap_1)(l-bp_2)},$$

or $$2lmn + l(bm-cn)p + m(cn-al)p_1 + n(al-bm)p_2$$
$$+ c(al+bm)pp_1 + a(bm+cn)p_1p_2 + q(cn+al)p_2p = 0 \;\dots\dots (6);$$

therefore, putting in this the values of p, p_1, p_2 derived from (2), (3), (4), the coefficients of $\alpha^2, \beta^2, \gamma^2$ will be found to vanish in the result; the other coefficients have the common factor $al+bm+cn$ which may therefore be omitted; and then the coefficient of $\beta\gamma$ is

$$l(bm+cn) = \tfrac{1}{2}a \cot A (\sin 2B + \sin 2C) = a \cos A \cos (B-C),$$

with similar expressions for the coefficients of $\gamma\alpha$ and $\alpha\beta$; hence the equation of the required locus is

$$\frac{a}{\alpha}\cos A \cos (B-C) + \frac{b}{\beta}\cos B \cos (C-A) + \frac{c}{\gamma}\cos C \cos (A-B) = 0.$$

2479. (Proposed by R. BALL, M.A.)—Determine m so that

$$\frac{-3b+am}{2a} + \frac{\sqrt{(9b^2-12ac-6abm-3a^2m^2)}}{2a}$$

shall be a root of the cubic $ax^3+3bx^2+3cx+d=0.$

Solution by SAMUEL ROBERTS, M.A.

A root of the cubic is of the form $-\dfrac{b+N}{a}$, where N is a known function of the coefficients. Comparing this with the given form, we have

$$\frac{m}{2} + \frac{b}{2a} + \frac{\sqrt{\phi}}{2a} = \frac{N}{a}$$

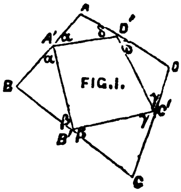
and therefore $$\phi = (am+b-2N)^2,$$

or $$4a^2m^2 + (8b-4N)\,m + (b-2N)^2 - 9b^2 + 12ae,$$

whence $$2am = -(2b-N) + \sqrt{\left\{12(b^2-ae) - 3N^2\right\}}.$$

2288. (Proposed by W. S. B. WOOLHOUSE, F.R.A.S.)—In Quest. 1705 *Reprint,* Vol. III., p. 109), it is stated, that "if from the intersection of the diagonals of a quadrilateral inscribed in a circle perpendiculars be drawn on the sides, the quadrilateral formed by joining the feet of these perpendiculars is of all quadrilaterals inscribed in the given one, the one of least perimeter." Discuss the more general case in which the given quadrilateral is not inscribable in a circle.

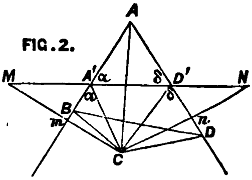
Solution by the PROPOSER.



It is somewhat remarkable that the inscription of a quadrilateral of minimum perimeter within a given quadrilateral is possible only when the given quadrilateral is inscribable in a circle. When the perimeter of an inscribed quadrilateral A′B′C′D′ is a minimum, its sides must make equal angles with each side of the given quadrilateral ABCD. Let these equal angles be denoted by α, β, γ, δ as marked in the diagram (Fig. 1) and we shall have these relations amongst the several angles, viz.,

$$A = \pi - (\alpha + \delta), \qquad B = \pi - (\alpha + \beta),$$
$$C = \pi - (\beta + \gamma), \qquad D = \pi - (\gamma + \delta).$$

Therefore $\quad A + C = B + D$; but $A + C + B + D = 2\pi$; whence $A + C = B + D = \pi$, and the quadrilateral is inscribable in a circle. What therefore becomes of the general case in which ABCD is not so inscribable? On examination it will be found that one side B′C′ of the inscribed quadrilateral vanishes, forming a double point at the angle C, and that the figure thus merges into a triangle CA′D′. Draw Cm (Fig. 2) perpendicular to AB and produce it to M, making Cm = mM; similarly draw CnN perpendicular to AD, making Cn = nN. Then M and N are reflected images of the angular point C with respect to the sides AB, AD. Join MN cutting these sides in A′, D′; and join CA′, CD′. Then CA′, A′D′ make equal angles (α) with AB, and CD′, A′D′ make equal angles (δ) with AD. The direction of the zero side which connects the double point at C is indeterminate, and may therefore satisfy any condition, but it is not competent to satisfy two conditions that are independent. The quadrilateral $A'\overline{CC}D'$, though not strictly of minimum perimeter according to analytical theory, is one of least perimeter, because the value of this perimeter would become increased by a small displacement of any one of the four points.



The perimeter $= CA' + A'D' + D'C = MA' + A'D' + D'N = MN$, which is the distance between the reflected images of C with respect to AB, AD, and is independent of the positions of the points B, D on those lines.

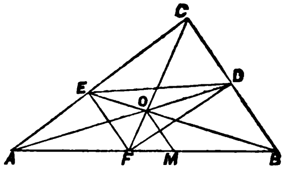
Again, since the angle $A'CD' = \pi - CA'D' - A'D'C = \pi - (\pi - 2\alpha) - (\pi - 2\delta)$ $= 2(\alpha + \delta) - \pi = \pi - 2A$, it follows that the construction will require the angle A to be acute; also that each acute angle of the given quadrilateral will determine a separate construction similar to that we have indicated.

COR.—When the given quadrilateral is inscribable in a circle in accordance with Quest. 1705, there are innumerable inscribed quadrilaterals of minimum perimeter, having their sides parallel and the perimeters equal to one another.

The construction we have given with a double point will be one of these, and will assist in determining the value of this perimeter, which according to what precedes is the distance of the reflected images of any angle with respect to the opposite sides. The distance of the reflected images is therefore of the same magnitude for each angle. For in this case we have the angle $MCN = BCD$; also $CM = 2BC \sin B$, and $CN = 2CD \sin D = 2CD \sin B$; therefore $MN = 2BD \sin B$, and if R denote the radius of the circumscribing circle, $R \, . \, MN = AC \, . \, BD$. That is, the rectangle under the radius and the minimum perimeter is equal to the rectangle under the diagonals of the given quadrilateral.

2537. (Proposed by the EDITOR.)—A point O is taken within a triangle ABC, and straight lines AOD, BOE, COF, are drawn through O, meeting the sides in D, E, F; show that the average area of the triangle DEF, in parts of the area of the triangle ABC, is $10-\pi^2$.

I. *Solution by* CAPTAIN A. R. CLARKE, R.E., F.R.S.



First to determine the average area of FDB. The following relations are obvious amongst the areas of the triangles,

$$\frac{\text{FDB}}{\text{DOB}}=\frac{\text{FC}}{\text{OC}},\quad \frac{\text{DOB}}{\text{AOB}}=\frac{\text{DO}}{\text{OA}},\quad \frac{\text{AOB}}{\text{ACB}}=\frac{\text{OF}}{\text{FC}}.$$

Multiplying these together we find that the area of FDB, expressed in parts of the area of triangle ABC ($=\Delta$), is

$$\text{FDB}=\frac{\text{FO}\,.\,\text{OD}}{\text{CO}\,.\,\text{OA}}\,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (a).$$

From O draw OM parallel to BC, meeting AB in M, and let AM $=x$ and MO $=y$ be the coordinates of O; then (a) becomes

$$\text{FDB}=\frac{y}{a-y}\,.\,\frac{c-x}{x}.$$

Consequently the average area of FDB is

$$\frac{1}{\Delta}\int_0^a\int_{\frac{cy}{a}}^c \frac{y}{a-y}\left(\frac{c}{x}-1\right)\sin\text{B}\,.\,dy\,dx$$

$$=2\int_0^1 \frac{\frac{y}{a}}{1-\frac{y}{a}}\left(-\log\frac{y}{a}-1+\frac{y}{a}\right)d\left(\frac{y}{a}\right)\ldots\ldots(\beta).$$

Now, by expanding $(1-\phi)^{-1}$ in series, it is easy to show that

$$\int_0^1 \frac{\phi}{1-\phi}\log\phi\,.\,d\phi=1-\frac{\pi^2}{6};$$

hence (β) becomes $=2\left(\frac{1}{6}\pi^2-1-\frac{1}{2}\right)$, and consequently the average area of the triangle DEF, in terms of the area of the original triangle ABC, is three times this quantity subtracted from unity, that is $10-\pi^2$, or ·1304 nearly.

[In the Editorial *Note* to the solution of Question 1683 (*Reprint*, Vol. V., p. 31) it is shown that the area of DEF, expressed in triangular coordinates (x, y, z), is $\dfrac{2xyz}{(1-x)(1-y)(1-z)}$; consequently the average area required is

$$2\int_0^1\int_0^{1-x} dx\,dy\,\frac{2xy(1-x-y)}{(1-x)(1-y)(x+y)}=4\int_0^1 dx\left(x+\frac{x^2\log x^2}{1-x^2}\right)\ldots(\gamma),$$

$$(\text{putting } x=\epsilon^{-z})=2-8\int_0^{\infty} dz\,.\,z\left(\epsilon^{-3z}+\epsilon^{-5z}+\epsilon^{-7z}+\ldots\ldots\right)\ldots.(\delta),$$

$$=2-8\left(\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots\ldots\right)=2-8\left(\frac{\pi^2}{8}-1\right)=10-\pi^2.$$

If the points D, E are taken arbitrarily, instead of the point O, Captain

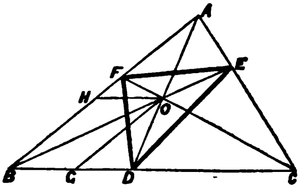
Clarke has found (see Quest. 2576, p. 52 of this volume) that the average area of the triangle DEF is $\frac{1}{18}(\pi^2-8)$, or ·1168 nearly.

If we proceed to the analogous problem in space of three dimensions, (viz., A point O is taken arbitrarily within a tetrahedron ABCD, and straight lines AOE, BOF, COG, DOH are drawn, meeting the opposite faces in E, F, G, H; to find the average volume of the tetrahedron EFGH,) we shall find, by the same method as that used in the *Note* above referred to, that the volume of EFGH, expressed in tetrahedral coordinates (w, x, y, z) is $\frac{3wxyz}{(1-w)(1-x)(1-y)(1-z)}$; hence the mean volume of EFGH, in parts of the volume of ABCD, will be given by the expression

$$6\int_0^1\int_0^{1-x}\int_0^{1-x-y} dx\,dy\,dz\,\frac{3xyz\,(1-x-y-z)}{(1-x)\,(1-y)\,(1-z)\,(x+y+z)}\;..\;(\epsilon),$$

which does not seem to admit of integration in finite terms.]

II. *Solution by* Stephen Watson.



Draw OG, OH respectively parallel to AB, BC, and put CG $= x$, AH $= y$, and triangle ABC $= \Delta$.

Then $\quad x : c-y = a :$ BF,

therefore $\quad \text{BF} = \dfrac{a\,(c-y)}{x}.$

Similarly $\quad \text{BD} = \dfrac{c\,(a-x)}{y}.$

Now an element of the triangle at O is $\sin B\, dx\, dy$; moreover the limits are y from $\frac{c\,(a-x)}{a}$ to c, and of x from 0 to a.

Hence the average area of the triangle BDF is

$$\frac{\sin^2 B \iint \text{BD}\,.\,\text{BF}\,.\,dx\,dy}{2\Delta} = \sin B\int_0^a dx \int_{\frac{c(a-x)}{a}}^{c} \frac{(a-x)\,(c-y)}{xy}\,dy$$

$$= \sin B\int_0^a \left(-\frac{cx}{a} - c\log\frac{a-x}{a}\right)\frac{a-x}{x}\,dx$$

$$= -\Delta + c\sin B\int_0^a \log\frac{a-x}{a}\,dx - 2\Delta\int_0^a \log\left(1-\frac{x}{a}\right)\frac{dx}{x}$$

$$= -3\Delta + 2\Delta\int_0^a \left\{\frac{x}{a} + \frac{1}{2}\left(\frac{x}{a}\right)^2 + \frac{1}{3}\left(\frac{x}{a}\right)^3 + \&c.\right\}\frac{dx}{x}$$

$$= -3\Delta + 2\Delta\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \&c.\right)$$

$$= -3\Delta + \frac{\pi^2}{3}\Delta.$$

The result will be the same for the average area of each of the triangles CDE, AEF; hence the average area of the triangle DEF is

$$\Delta - 3\left(-3\Delta + \frac{\pi^2}{3}\Delta\right) = (10-\pi^2)\,\Delta.$$

III. *Solution by* SAMUEL BILLS.

1. For the plane problem enunciated in the question.

Referring to the diagram of the foregoing Solution, and putting $BG = x$, $GO = y$, the area of the triangle DEF, in parts of the area of ABC, is

$$\frac{\text{area DEF}}{\text{area ABC}} = \frac{2xy\,(ac-ay-cx)}{(a-x)\,(c-y)\,(ay+cx)}, \text{ (*Reprint*, Vol. V., p. 29) } \ldots(A).$$

Now an element at O is $\sin B\, dx\, dy$; also the measure of the number of positions of O, that is to say, of the total number of inscribed triangles, is $\triangle ABC = \frac{1}{2} ac \sin B$; hence the average required is

$$\frac{4}{ac}\int_0^a\int_0^{\frac{c}{a}(a-x)} dx\, dy\, \frac{xy\,(ac-ay-cx)}{(a-x)\,(c-y)\,(ay+cx)} \ldots\ldots\ldots\ldots(B).$$

Integrating with respect to y, (B) becomes

$$\frac{4}{a}\int_0^a dx \left(\frac{x}{a} + \frac{2x^2}{a^2-x^2}\log\frac{x}{a}\right) \ldots\ldots\ldots\ldots\ldots\ldots(C).$$

[Mr. BILLS integrates (C) by a rather long process; if, however, we put $x = a\epsilon^{-z}$, the expression (C) will reduce to the form marked (δ) in the Editorial *Note* to Capt. CLARKE's Solution.]

2. For the analogous problem in regard to a tetrahedron, enunciated in the Editorial *Note* at the end of the first Solution.

Take the vertex A of the tetrahedron as origin of Cartesian coordinates, and the edges AB ($=b$), AC ($=c$), AD ($=d$) as axes of x, y, z respectively; and put (x, y, z) for the coordinates of O, (α, β, γ) for the plane angles meeting at the solid angle O, and

$$m = (1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma)^{\frac{1}{2}};$$

then, finding the coordinates of E, F, G, H in the usual way, and substituting their values in the formula for the volume of a tetrahedron in terms of the coordinates of its vertices, we find that the volume of the inscribed tetrahedron EFGH, expressed in parts of the volume of ABCD ($=\frac{1}{6}mbcd$) is

$$\frac{\text{vol. EFGH}}{\text{vol. ABCD}} = \frac{3xyz\,(bcd-cdx-dby-bcz)}{(b-x)\,(c-y)\,(d-z)\,(cdx+dby+bcz)} \ldots\ldots\ldots(D).$$

Now an element at O is $m\, dx\, dy\, dz$; also the measure of the number of positions of O is vol. ABCD $= \frac{1}{6}mbcd$; hence the average volume of the tetrahedron EFGH, in parts of the volume of ABCD, will be given by the expression

$$\frac{6}{bcd}\int_0^b\int_0^{c\left(1-\frac{x}{b}\right)}\int_0^{d\left(1-\frac{x}{b}-\frac{y}{c}\right)} dx\, dy\, dz\, \frac{3xyz\,(bcd-cdx-dby-bcz)}{(b-x)\,(c-y)\,(d-z)\,(cdx+dby+bcz)}$$
$$\ldots\ldots\ldots\ldots(E).$$

I have tried this integral by a variety of methods, but have not been able to find its value in any way.

[If in Mr. BILLS's investigation we suppose the triangle ABC to have the sides BA and BC each equal to unity, as we may do without loss of generality, the expressions (B) and (C) become (by making $a = c = 1$) identical with those marked (γ) in the *Note* at the end of the first Solution. Similarly in the tetrahedron, if we suppose the edges AB, AC, AD to be each equal to unity, the expression (E) becomes (by making in it $b=c=d=1$) identical with that marked (ϵ) in the *Note* to the first Solution.]

2576. (Proposed by Captain **Clarke**, F.R.S.)—If in Question 2537 the points D, E be taken arbitrarily, instead of the point O, show that the average area of the triangle DEF, in parts of the area of ABC, is $\frac{1}{16}(\pi^2-8)$.

Solution by the Rev. Joseph Wolstenholme, M.A.

Referring to the diagram in the solution to Question 2537 (see p. 49 of this volume), let $\frac{\mathrm{BD}}{\mathrm{BC}}=x$, $\frac{\mathrm{CE}}{\mathrm{CA}}=y$, $\frac{\mathrm{AF}}{\mathrm{AB}}=z$; then $xyz=(1-x)(1-y)(1-z)$, therefore the area of the triangle DEF in parts of the area of ABC, is

$$\Delta\,\mathrm{DEF}=1-x(1-z)-y(1-x)-z(1-y)=\frac{2xy(1-x)(1-y)}{1-x-y+2xy}\ldots\ldots(\alpha).$$

Now x and y being equally likely to have any value from 0 to 1, the average value of (α) is $\int_0^1\int_0^1(\alpha)\,dx\,dy$, which, if we put $2x=1+x'$ and $2y=1+y'$, becomes, suppressing the unnecessary accents,

$$\frac{1}{16}\int_{-1}^{+1}\int_{-1}^{+1}\frac{(1-x^2)(1-y^2)}{1+xy}\,dx\,dy$$

$$=\frac{1}{8}\int_{-1}^{1}\frac{1-x^2}{x^2}\left(1-\frac{1-x^2}{2x}\log\frac{1+x}{1-x}\right)dx\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\beta)$$

$$=\frac{1}{8}\int_{-1}^{1}\frac{1-x^2}{x^3}\left\{x-(1-x^2)\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots\right)\right\}dx$$

$$=\frac{1}{8}\int_{-1}^{1}\left\{(1-\tfrac{1}{3})-x^2(1-\tfrac{2}{3}+\tfrac{1}{5})-x^4(\tfrac{1}{3}-\tfrac{2}{5}+\tfrac{1}{7})-\ldots\right\}dx$$

$$=\frac{1}{4}\left\{(1-\tfrac{1}{3})-\tfrac{1}{3}(1-\tfrac{2}{3}+\tfrac{1}{5})-\tfrac{1}{5}(\tfrac{1}{3}-\tfrac{2}{5}+\tfrac{1}{7})-\ldots\right\}$$

$$=\frac{1}{2}\left(\frac{1}{3^2}+\frac{1}{5^2}+\ldots\right)+\frac{1}{4}\left(1-\frac{2}{1.3}-\frac{2}{3.5}-\frac{2}{5.7}-\ldots=0\right)=\frac{\pi^2-8}{16}.$$

[Capt. **Clarke** and Mr. **Watson** integrate the expression (β) in the following manner. Since this function of x remains unchanged when $-x$ is written for x, it will be sufficient to integrate from $x=0$ to $x=1$, and double the result; thus (β) becomes

$$\frac{1}{4}\int_0^1 \left(\frac{1-x^2}{x^2}-\frac{(1-x^2)^2}{2x^3}\log\frac{1+x}{1-x}\right)dx = \frac{1}{8}\left[\frac{1-x^4}{2x^3}\log\frac{1+x}{1-x}-\frac{1}{x}-3x\right]_0^1$$
$$+\frac{1}{4}\int_0^1 \frac{1}{x}\log\frac{1+x}{1-x}\,dx.$$

Now $\displaystyle\frac{1-x^4}{2x^3}\log\frac{1+x}{1-x}-\frac{1}{x}=\frac{x}{3}-4\left(\frac{x^3}{1.5}+\frac{x^5}{3.7}+\frac{x^7}{5.9}+\dots\right)$,

which is -1 when x is unity, and 0 when x is zero. Also

$$\int_0^1 \frac{1}{x}\log\frac{1+x}{1-x}\,dx = 2\int_0^1\left(1+\frac{x^2}{3}+\frac{x^5}{5}+\dots\right)dx = \frac{\pi^2}{4};$$

hence the average required is $\frac{1}{16}(\pi^2-8)$.]

2575. (Proposed by C. W. MERRIFIELD, F.R.S.)—Given two surfaces of the second degree, there exists *in general** a set of Cartesian axes, whose directions are those of conjugate diameters in every one of the surfaces of the second degree passing through the intersection* of the two surfaces given. The directions will be those of the three paraboloids* of the system of surfaces. The locus of centres of the system lies on the intersection of three cylinders of the second degree whose axes are parallel to those of the paraboloids of the system.

I. *Solution by* Professor TAIT.

A solution of this question is readily obtained by *Quaternions*; at least as regards *central* surfaces, and it is quite easy to extend it to the others. For any surface through the intersection of

$S\rho\phi\rho = 1$ and $S(\rho-\alpha)\,\psi\,(\rho-\alpha) = e$, is $kS\rho\phi\rho - S(\rho-\alpha)\,\psi\,(\rho-\alpha) = k-e$,

where k and e are scalars.

The axes of this depend only on the term $S\rho\,(k\phi-\psi)\,\rho$.
Hence the set of conjugate diameters which are the same in all are the roots of

$$V\,(k\phi-\psi)\,\rho\,(k_1\phi-\psi)\,\rho = 0, \text{ or } V\phi\rho\psi\rho = 0,$$

as we might have seen without analysis.

The locus of the centres is given by the equation

$$(\psi-k\phi)\,\rho-\psi\alpha = 0,$$

where k is a scalar variable.

These equations give the required results at once.

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* Real or imaginary: the reality of the reductions is another matter.

II. *Solution by the* Rev. Joseph Wolstenholme, M.A.

Let the equation of one surface be

$$ax^2+by^2+cz^2=1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1);$$

and let one be described similar, and similarly situated to the second, but concentric with this; and let the equation of this be

$$Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=1 \quad \ldots\ldots\ldots\ldots (2).$$

Now, if a plane $lx+my+nz=0$ be taken, the conjugate diameters with respect to (1) and (2) are

$$\frac{ax}{l}=\frac{by}{m}=\frac{cz}{n}; \quad \text{and} \quad \frac{Ax+Hy+Gz}{l}=\frac{Hx+By+Fz}{m}=\frac{Gx+Fy+Cz}{m}.$$

If these coincide, and (λ, μ, ν) be the direction, we have

$$A\lambda+H\mu+G\nu=ka\lambda, \text{ \&c.},$$

giving the cubic $\begin{vmatrix} A-ka, & H, & G \\ H, & B-kb, & F \\ G, & F, & C-kc \end{vmatrix}=0$; which, when developed, is

$$(A-ka)(B-kb)(C-kc)-F^2(A-ka)-G^2(B-kb)-H^2(C-kc)+2FGH=0.$$

This may be written in the form

$$(k-u)(k-v)(k-w)-\frac{GH}{aF}(k-v)(k-w)-\ldots\ldots=0;$$

where $\quad A-\dfrac{GH}{F}=au, \quad B-\dfrac{HF}{G}=bv, \quad C-\dfrac{FG}{H}=cw,$

and it can be readily shown to have three real roots, if abcFGH be positive; but this is not a criterion.

Supposing k_1, k_2, k_3 to be the roots, and $(\lambda_1, \mu_1, \nu_1)$, &c. the corresponding values of (λ, μ, ν); then $A\lambda_1+H\mu_1+G\nu_1=k_1a\lambda_1$, &c.

Multiplying by λ_2, μ_2, ν_2, and adding, we get

$$\lambda_1(A\lambda_2+H\mu_2+G\nu_2)+\ldots=k_1(a\lambda_1\lambda_2+b\mu_1\mu_2+c\nu_1\nu_2), \text{ or } (k_1-k_2)(a\lambda_1\lambda_2+\ldots)=0;$$

or except when the above cubic has equal roots, the three directions corresponding to the three roots form a system of conjugate diameters for (1), and similarly for (2). We get therefore for any two conicoids one system of axes parallel to a system of conjugate diameters for both, and referred to these the equations of the two will be

$$u\equiv \alpha x^2+\beta y^2+\gamma z^2-1=0, \quad v\equiv \alpha' x^2+\ldots+2\alpha'' x+\ldots=0;$$

and the general equation of a conicoid through their intersections is $u+rv=0$; hence any such conicoid has a system of conjugate diameters parallel to the axes of coordinates. Also, if we eliminate successively x^2, y^2, z^2, we shall get the equations of three paraboloids through the curve of intersection, and their axes are parallel to the axes of coordinates.

The locus of centres is $\dfrac{du}{dx}\div\dfrac{dv}{dx}=\dfrac{du}{dy}\div\dfrac{dv}{dy}=\dfrac{du}{dz}\div\dfrac{dv}{dx}$; and any one of these equations gives a hyperbolic cylinder with its axis parallel to one of the coordinate axes.

III. *Solution by the* Proposer.

The two surfaces may be written, without any loss of generality upon their intersection, as

$$U=a_1x^2+b\,.\,y^2+c_1z^2+1=0.$$
$$V=a_2(x-\alpha)^2+b_2(y-\beta)^2+c_2(z-\gamma)^2+1=0.$$

For these contain nine independent constants, and there are nine more involved in the choice of axes;—viz., 3 for origin, 3 for inclination of axes, and 3 for aspect, making 18 in all, which is just the number in two quaternary quadrics of the most general form.

Now, $a_2U-a_1V=0$, $b_2U-b_1V=0$, $c_2U-c_1V=0$ are three paraboloids of which the axes are parallel to those of x, y, and z. The form of the equations shows that the axes are parallel to conjugate diameters; and, k being indeterminate, $U+kV=0$ is of the same form as V, becoming parabolic when

$$k=-\frac{a_1}{a_2},\ -\frac{b_1}{b_2},\ \text{or}\ -\frac{c_1}{c_2}.$$

Again, the coordinates (x, y, z) of the centre of $U+kV=0$ are

$$x=\frac{a_2\alpha k}{a_1+ka_2},\quad y=\frac{b_2\beta k}{b_1+kb_2},\quad z=\frac{c_2\gamma k}{c_1+kc_2};$$

therefore

$$k=\frac{a_1x}{a_2(\alpha-x)}=\frac{b_1y}{b_2(\beta-y)}=\frac{c_1z}{c_2(\gamma-z)},$$

which is the intersection of three cylinders of the second degree, written in hyperbolic form, whose axes are parallel to those of reference.

2553. (Proposed by Professor CAYLEY.)—Show that the surface $y^2z^2+z^2x^2+x^2y^2-2xyz=0$ meets the sphere $x^2+y^2+z^2=1$ in four circles; and explain in a general manner the form of the curve of intersection of the surface by any other sphere having the same centre, and thence the form of the surface itself (being a particular case of Steiner's surface, and which by the homographic transformations $w^{-1}x$, $w^{-1}y$, $w^{-1}z$, for x, y, z gives $y^2z^2+z^2x^2+x^2y^2-2wxyz=0$, the general equation of Steiner's surface).

Solution by the PROPOSER.

Take X, X′, Y, Y′, Z, Z′ the intersections of the sphere $x^2+y^2+z^2=1$ by the three axes respectively; then we have $x^2+y^2+z^2=1$, $x+y+z=-1$, the equations of the circle through the points X′, Y′, Z′; and from these two equations we deduce $yz+zx+xy=0$, and thence

$$y^2z^2+z^2x^2+x^2y^2+2xyz\,(x+y+z)=0,$$

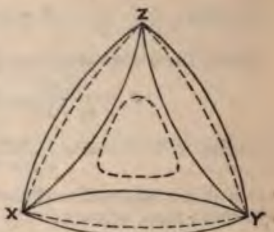
that is

$$y^2z^2+z^2x^2+x^2y^2-2xyz=0;$$

so that the circle lies on the quartic surface; and by changing successively the signs of each two of the three coordinates, we have three other circles lying on the sphere and also on the quartic surface; viz., we have in all four circles, the above mentioned circle through (X′, Y′, Z′), and three other circles through (X′, Y, Z), (X, Y′, Z), (X, Y, Z′) respectively, making together a curve of the order 8, the complete intersection of the quartic surface by the sphere.

The quartic surface lies entirely in the four octants of space xyz, $xy'z'$, $x'yz'$, $x'y'z$; and as to the portion of the surface which lies in the octant xyz, this meets the sphere $x^2+y^2+z^2=1$ in portions of the three circles (X′, Y, Z) (X, Y′, Z) (X, Y, Z′) constituting a tricuspidal form lying within the octant XYZ as shown in the figure. The intersection by a sphere, radius

<1, projected on the octant XYZ, is a trinodal form, lying outside the tricuspidal one, as shown by a dotted line in the figure; the intersection by a sphere radius > 1, projected in the same way, is a trigonoid form lying inside the tricuspidal one, as also shown by a dotted line in the figure; as the radius approaches to and ultimately becomes $= \frac{2}{\sqrt{3}}$, this diminishes, and becomes ultimately a mere point, and when the radius is greater than this value the intersection is imaginary.



Imagine on the solid sphere, radius $= 1$, the four tricuspidal forms lying in alternate octants as above; cut away down to the centre the portions lying without these tricuspidal forms; and build up on the tricuspidal forms, until the greatest distance from the centre becomes $= \frac{2}{\sqrt{3}}$; we have a solid figure with four prominences situate as the summits of a tetrahedron, the bounding surface whereof is the surface in question: it is to be added that the axes are nodal lines on the surface, viz., the portions which lie within the solid figure are the intersections of two real sheets of the surface, the portions which lie without the solid figure are isolated, or acnodal, lines on the surface.

2481. (Proposed by the Rev. J. Blissard.)—Find the developments of $\frac{(\sin^{-1} x)^{2n-1}}{\Gamma(2n)}$ and $\frac{(\sin^{-1} x)^{2n}}{\Gamma(2n+1)}$, and show that they possess a property analogous to that noticed by Professor Sylvester (see Question 2447)* as belonging to the developments of $(\sin^{-1} x)$ and $\frac{1}{2}(\sin^{-1} x)^2$.

Solution by the Proposer.

The required developments may be arrived at either by an inductive or a deductive method. The former being the simpler, we shall use it here, and proceed step by step.

[illegible] $= P_1 \sin x + P_3 \frac{\sin^3 x}{1.2.3} + \ldots + P_{2n-1} \frac{\sin^{2n-1} x}{1.2\ldots(2n-1)} + \ldots.$

Then we have

[illegible] $\frac{\sin^2 x}{1.2} + \ldots\Big) + (1-\sin^2 x)\left(P_3 \sin x + P_5 \frac{\sin^3 x}{1.2.3} + \ldots\right);$

[illegible] $\frac{[illegible]}{(2n-1)} - P_{2n-1}\left(\frac{1}{1.2\ldots(2n-2)} + \frac{1}{1.2\ldots(2n-3)}\right) = 0;$

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* *Reprint*, Vol. VIII., p. 59.

therefore $P_{2n+1} = (2n-1)^2 P_{2n-1} = (2n-1)^2 (2n-3)^2 P_{2n-3}\ldots$

$= (2n-1)^2 (2n-3)^2 \ldots 3^2 . 1^2 . P_1$;

but $P_1 = 1$, hence we have

$$\frac{P_{2n+1}}{1.2\ldots(2n+1)} = \frac{1^2.3^2\ldots(2n-1)^2}{1.2\ldots(2n+1)} = \frac{1.3\ldots(2n-1)}{2.4\ldots 2n}\cdot\frac{1}{2n+1}.$$

Hence $$x = \sin x + \frac{1}{2}\cdot\frac{\sin^3 x}{3} + \frac{1.3}{2.4}\cdot\frac{\sin^5 x}{5} + \ldots;$$

and putting $\sin^{-1} x$ for x, $\sin^{-1} x = \frac{x}{1} + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \ldots.$

2. Let $$\frac{x^2}{1.2} = P_2\frac{\sin^2 x}{1.2} + P_4\frac{\sin^4 x}{1.2.3.4} + \ldots.$$

Differentiating twice, we have

$$1 = -\sin x\left(P_2\frac{\sin x}{1} + P_4\frac{\sin^3 x}{1.2.3} + \ldots\right) + (1-\sin^2 x)\left(P_2 + P_4\frac{\sin^2 x}{1.2} + \ldots\right).$$

Hence $P_{2n+2} = (2n)^2 P_{2n} = (2n)^2 (2n-2)^2 \ldots 4^2 . 2^2 . P_2$; but $P_2 = 1$,

therefore $$\frac{P_{2n+2}}{1.2\ldots(2n+2)} = \frac{2^2.4^2\ldots(2n)^2}{1.2\ldots(2n+2)} = \frac{2.4\ldots(2n)}{3.5\ldots(2n+1)}\cdot\frac{1}{2n+2},$$

therefore $$\frac{x^2}{1.2} = \frac{\sin^2 x}{2} + \frac{2}{3}\cdot\frac{\sin^4 x}{4} + \frac{2.4}{3.5}\cdot\frac{\sin^6 x}{6} + \ldots,$$

and $$\frac{(\sin^{-1} x)^2}{1.2} = \frac{x^2}{2} + \frac{2}{3}\cdot\frac{x^4}{4} + \frac{2.4}{3.5}\cdot\frac{x^6}{6} + \ldots.$$

3. Let $$\frac{x^3}{1.2.3} = P_3\frac{\sin^3 x}{1.2.3} + P_5\frac{\sin^5 x}{1.2\ldots5} + \ldots.$$

Differentiating twice, and substituting $\sin^{-1} x$ for x, we have

$$\sin^{-1} x = -x\left(P_3\frac{x^2}{1.2} + P_5\frac{x^4}{1.2.3.4} + \ldots\right) + (1-x^2)\left(P_3 x + P_5\frac{x^3}{1.2.3} + \ldots\right);$$

but $\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \&c.$; therefore, equating coefficients of x^{2n+1},

$$\frac{P_{2n+3}}{1.2\ldots(2n+1)} - P_{2n+1}\left(\frac{1}{1.2\ldots 2n} + \frac{1}{1.2\ldots(2n-1)}\right) = \frac{1.3\ldots(2n-1)}{2.4\ldots 2n}\cdot\frac{1}{2n+1},$$

therefore $$P_{2n+3} = (2n+1)^2 P_{2n+1} + 1^2.3^2\ldots(2n+1)^2\cdot\frac{1}{(2n+1)^2},$$

and $$P_{2n+1} = (2n-1)^2 P_{2n-1} + 1^2.3^2\ldots(2n-1)^2\cdot\frac{1}{(2n-1)^2},$$

$$\therefore P_{2n+3} = (2n+1)^2(2n-1)^2 P_{2n-1} + 1^2.3^2\ldots(2n+1)^2\left(\frac{1}{(2n+1)^2} + \frac{1}{(2n-1)^2}\right),$$

$$\therefore P_{2n+3} = (2n+1)^2(2n-1)^2\ldots 3^2 P_3$$
$$+ 1^2.3^2\ldots(2n+1)^2\left(\frac{1}{(2n+1)^2} + \frac{1}{(2n-1)^2} + \ldots + \frac{1}{3^2}\right).$$ But $P_3 = 1$,

therefore $$P_{2n+3} = 1^2.3^2...(2n+1)^2\left(\frac{1}{1^2}+\frac{1}{3^2}+...+\frac{1}{(2n+1)^2}\right);$$

$$\therefore \frac{P_{2n+3}}{1.2...(2n+3)} = \frac{1.3...(2n+1)}{2.4...(2n+2)}\cdot\frac{1}{2n+3}\left(\frac{1}{1^2}+\frac{1}{3^2}+...+\frac{1}{(2n+1)^2}\right).$$

Hence $$\frac{(\sin^{-1}x)^3}{1.2.3} = \frac{1}{2}\cdot\frac{1}{1^2}\frac{x^3}{3}+\frac{1.3}{2.4}\left(\frac{1}{1^2}+\frac{1}{3^2}\right)\frac{x^5}{5}+....$$

4. Let $$\frac{x^4}{1.2.3.4} = P_4\frac{\sin^4 x}{1.2.3.4}+P_6\frac{\sin^6 x}{1.2...6}+...,$$

then by an exactly similar process we obtain

$$\frac{P_{2n+4}}{1.2...(2n+4)} = \frac{2.4...(2n+2)}{3.5...(2n+3)}\cdot\frac{1}{2n+4}\left(\frac{1}{2^2}+\frac{1}{4^2}+...+\frac{1}{(2n+2)^2}\right),$$

therefore $$\frac{(\sin^{-1}x)^4}{1.2.3.4} = \frac{2}{3}\cdot\frac{1}{2^2}\cdot\frac{x^4}{4}+\frac{2.4}{3.5}\left(\frac{1}{2^2}+\frac{1}{4^2}\right)\frac{x^6}{6}+....$$

5. Let $$\frac{x^5}{1.2.3.4.5} = P_5\frac{\sin^5 x}{1.2...5}+P_7\frac{\sin^7 x}{1.2...7}+...,$$

then differentiating twice, and putting $\sin^{-1}x$ for x, we obtain

$$\frac{(\sin^{-1}x)^3}{1.2.3} = -x\left(P_5\frac{x^4}{1.2.3.4}+P_7\frac{x^6}{1.2...6}+...\right)+(1-x^2)\left(P_5\frac{x^3}{1.2.3}+...\right).$$

But $$\frac{(\sin^{-1}x)^3}{1.2.3} = \frac{1}{2}\cdot\frac{1}{1^2}\cdot\frac{x^3}{3}+\frac{1.3}{2.4}\left(\frac{1}{1^2}+\frac{1}{3^2}\right)\frac{x^5}{5}+...;$$

hence equating coefficients of x^{2n+3}, we have, after slight reduction,

$$P_{2n+5} = (2n+3)^2P_{2n+3}+1^2.3^2...(2n+1)^2\left(\frac{1}{1^2}+\frac{1}{3^2}+...+\frac{1}{(2n+1)^2}\right).$$

Denote $\frac{1}{1^2}+\frac{1}{3^2}+...\frac{1}{(2n+1)^2}$ by C_{2n+1}, then we have

$$P_{2n+5} = (2n+3)^2P_{2n+3}+1^2.3^2...(2n+3)^2.\frac{C_{2n+1}}{(2n+3)^2},$$

$$= (2n+3)^2(2n+1)^2P_{2n+1}+1^2.3^2...(2n+3)^2\left(\frac{C_{2n+1}}{(2n+3)^2}+\frac{C_{2n-1}}{(2n+1)^2}\right),$$

$$= (2n+3)^2(2n+1)^2...3^2P_3$$
$$+1^2.3^2...(2n+3)^2\left(\frac{C_1}{3^2}+\frac{C_3}{5^2}+...+\frac{C_{2n+1}}{(2n+3)^2}\right).$$ But $P_3=0$,

$$\therefore \frac{P_{2n+5}}{1.2...(2n+5)} = \frac{1.3...(2n+3)}{2.4...(2n+4)}\cdot\frac{1}{2n+5}\left(\frac{C_1}{3^2}+\frac{C_3}{5^2}+...+\frac{C_{2n+1}}{(2n+3)^2}\right),$$

$$\therefore \frac{(\sin^{-1}x)^5}{1.2.3.4.5} = \frac{1.3}{2.4}\cdot\frac{C_1}{3^2}\cdot\frac{x^5}{5}+\frac{1.3.5}{2.4.6}\cdot\left(\frac{C_1}{3^2}+\frac{C_3}{5^2}\right)\cdot\frac{x^7}{7}+...$$

$$=\frac{1.3}{2.4}\cdot\frac{1}{1^2.3^2}\cdot\frac{x^5}{5}+\frac{1.3.5}{2.4.6}\left\{\frac{1}{1^2.3^2}+\frac{1}{5^2}\left(\frac{1}{1^2}+\frac{1}{3^2}\right)\right\}\frac{x^7}{7}+.$$

6. Let $$\frac{x^6}{1.2\ldots6} = P_6\frac{\sin^6 x}{1.2\ldots6} + P_8\frac{\sin^8 x}{1.2\ldots8} + \ldots,$$

then by a similar process, if $C_{2n} = \frac{1}{2^2} + \frac{1}{4^2} + \ldots + \frac{1}{(2n)^2}$, we obtain

$$\frac{(\sin^{-1}x)^6}{1.2\ldots6} = \frac{2.4}{3.5}\cdot\frac{C_2}{4^2}\cdot\frac{x^6}{6} + \frac{2.4.6}{3.5.7}\left(\frac{C_2}{4^2} + \frac{C_4}{6^2}\right)\frac{x^8}{8} + \ldots$$

$$= \frac{2.4}{3.5}\cdot\frac{1}{2^2.4^2}\cdot\frac{x^6}{6} + \frac{2.4.6}{3.5.7}\left\{\frac{1}{2^2.4^2} + \frac{1}{6^2}\left(\frac{1}{2^2} + \frac{1}{4^2}\right)\right\}\frac{x^8}{8} + \ldots.$$

7. The forms which the expansions of $\frac{(\sin^{-1}x)^{2n-1}}{1.2\ldots(2n-1)}$ and $\frac{(\sin^{-1}x)^{2n}}{1.2\ldots(2n)}$ assume now become evident. Thus, let

$$\frac{1}{1^2} + \frac{1}{3^2} + \ldots + \frac{1}{(2n-1)^2} = \phi_1(2n-1),$$

$$\frac{1}{3^2}\phi_1(1) + \frac{1}{5^2}\phi_1(3) + \ldots + \frac{1}{(2n+1)^2}\phi_1(2n-1) = \phi_3(2n-1),$$

$$\frac{1}{5^2}\phi_3(1) + \frac{1}{7^2}\phi_3(3) + \ldots + \frac{1}{(2n+3)^2}\phi_3(2n-1) = \phi_5(2n-1),\ \&c.;\ \text{then}$$

$$\frac{(\sin^{-1}x)^{2n-1}}{\Gamma(2n)} = \frac{1.3\ldots(2n-3)}{2.4\ldots(2n-2)}\phi_{2n-3}(1)\cdot\frac{x^{2n-1}}{2n-1}$$
$$+ \frac{1.3\ldots(2n-1)}{2.4\ldots2n}\cdot\phi_{2n-3}(3)\cdot\frac{x^{2n+1}}{2n+1} + \ldots\ldots\ldots\ldots(\text{I.})$$

Also if $$\frac{1}{2^2} + \frac{1}{4^2} + \ldots + \frac{1}{(2n)^2} = \phi_2(2n),$$

$$\frac{1}{4^2}\phi_2(2) + \frac{1}{6^2}\phi_2(4) + \ldots + \frac{1}{(2n+2)^2}\phi_2(2n) = \phi_4(2n),$$

$$\frac{1}{6^2}\phi_4(2) + \frac{1}{8^2}\phi_4(4) + \ldots + \frac{1}{(2n+4)^2}\phi_4(2n) = \phi_6(2n),\ \&c.;\ \text{then}$$

$$\frac{(\sin^{-1}x)^{2n}}{\Gamma(2n+1)} = \frac{2.4\ldots(2n-2)}{3.5\ldots(2n-1)}\phi_{2n-2}(2)\cdot\frac{x^{2n}}{2n}$$
$$+ \frac{2.4\ldots2n}{3.5\ldots(2n+1)}\cdot\phi_{2n-2}(4)\cdot\frac{x^{2n+2}}{2n+2} + \ldots\ldots\ldots\ldots(\text{II.})$$

It will be perceived that the property noticed by Professor Sylvester holds good in the above expansions. The formula (II.) may be derived from (I.) by augmenting in it every factor, every index, and every sub-index, by unity.

When $n=1$, the above formulæ (I.) and (II.) may appear to fail, as not giving immediately the expansions of $\sin^{-1}x$ and $\frac{(\sin^{-1}x)^2}{1.2}$. They are, however, perfectly correct, and we may obtain the required expansions from them as follows:—

$$\frac{(\sin^{-1}x)^{2n-1}}{\Gamma(2n)} = \frac{\Gamma(n-\frac{1}{2})}{\Gamma(\frac{1}{2})\,\Gamma(n)}\cdot\phi_{2n-3}(1)\cdot\frac{x^{2n-1}}{2n-1}$$
$$+\frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})\,\Gamma(n+1)}\cdot\phi_{2n-3}(3)\cdot\frac{x^{2n+1}}{2n+1}+\dots,$$

$$\frac{(\sin^{-1}x)^{2n}}{\Gamma(2n+1)} = \frac{1}{2}\left\{\frac{\Gamma(\frac{1}{2})\,\Gamma(n)}{\Gamma(n+\frac{1}{2})}\cdot\phi_{2n-2}(2)\cdot\frac{x^{2n}}{2n}\right.$$
$$\left.+\frac{\Gamma(\frac{1}{2})\,\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}\cdot\phi_{2n-2}(4)\cdot\frac{x^{2n+2}}{2n+2}+\dots\right\},$$

$$\therefore\ (n=1),\ \frac{\sin^{-1}x}{\Gamma(2)} = \phi_{-1}(1)\cdot\frac{x}{1}+\frac{1}{2}\phi_{-1}(3)\cdot\frac{x^3}{3}+\frac{1.3}{2.4}\phi_{-1}(5)\cdot\frac{x^5}{5}+\dots(\alpha),$$

and $$\frac{(\sin^{-1}x)^2}{\Gamma(3)} = \phi_0(2)\cdot\frac{x^2}{2}+\frac{2}{3}\phi_0(4)\cdot\frac{x^4}{4}+\frac{2.4}{3.5}\phi_0(6)\cdot\frac{x^6}{6}+\dots(\beta).$$

Now $\phi_1(2n-1)\left(\text{which} = \frac{1}{1^2}+\frac{1}{3^2}+\dots+\frac{1}{(2n-1)^2}\right)$

$$= \frac{1}{1^2}\phi_{-1}(1)+\frac{1}{3^2}\cdot\phi_{-1}(3)+\dots+\frac{1}{(2n-1)^2}\phi_{-1}(2n-1).$$

Hence $\phi_{-1}(1)$, $\phi_{-1}(3)$, &c., are all equal to 1. Also, $\phi_2(2n)$ $\left(\text{which} = \frac{1}{2^2}+\frac{1}{4^2}+\dots+\frac{1}{(2n)^2}\right) = \frac{1}{2^2}\phi_0(2)+\frac{1}{4^2}\phi_0(4)+\dots+\frac{1}{(2n)^2}\phi_0(2n)$.

Hence $\phi_0(2)$, $\phi_0(4)$, &c., all $= 1$. Making these substitutions in (α) and (β), we obtain the expansions of $\sin^{-1}x$ and $\frac{(\sin^{-1}x)^2}{1.2}$.

NOTE.—In proposing Quest. 2236 (see *Reprint*, Vol. VII., p. 62), it did not occur to me that the formula (I.) is substantially the same as the well-known one $x = \sin x + \frac{1}{2}\cdot\frac{\sin^3 x}{3} + \text{\&c.}$ I would therefore wish to withdraw the statement, made at the end of Art. 1 of the solution, that the series there given for the value of π is probably new, which is not likely to be the case, since it may be obtained from the formula $x = \sin x + \text{\&c.}$, by making $x = \frac{1}{2}\pi$.

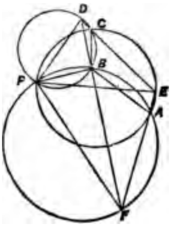
2441. (Proposed by F. A. TARLETON, M.A.)—Find the envelope of a conic which is circumscribed to a given triangle, and with respect to which two given lines are conjugate.

Solution by Professor HIRST.

Since a conic is cut harmonically by two conjugate lines, this Question is identical with the Question 1481, proposed by me and solved by Mr. GREER (*Reprint*, Vol. V., p. 85.)

2507. (Proposed by J. Wilson.)—Three circles pass through a point P; and PD, PE, PF are their diameters; if A, B, C be their other intersections, prove that the three triads of points EAF, FBD, DCE are collinear; and if a circle pass through the points P, D, E, F, the triad ABC is also collinear.

Solutions by Archer Stanley; R. Tucker, M.A.; *and others.*



1. Inverting the figure relative to the point P, we have the following well known and easily demonstrated theorems:—

"If D′E′F′ be the feet of the perpendiculars from a point P on the sides B′C, C′A′, A′B′ of a triangle, PE′A′F′, PF′B′D′, PD′C′E′ are three sets of concyclic points; and if moreover D′, E′, F′ are collinear, PA′B′C′ are also concyclic."

2. *Otherwise*:—The angle PAE = a right angle = PAF; therefore EAF is a straight line, and the like may be shown of FBD, DCE. Moreover if a circle pass through PDEF, the angles PDE, PFE are supplementary, and therefore PBC, PBA are also supplementary; hence, in this case, ABC will be a straight line.

[This Question might be stated more generally thus:—Three circles pass through a point P, and the chords PD, PE, PF subtend similar segments in the same direction, &c. According to a known theorem (See *Lady's and Gentleman's Diary*, Quest. 1934), if a point be taken on each of the sides of a triangle, the circles circumscribing the three triangles cut off by lines joining those points two and two, will intersect in one point; and the lines joining the common intersection to the points on the triangle make equal angles, in the same direction, with the sides; the evident converse of this gives the solution of the first part of the question. The second part, as stated in the question, was a Cambridge Senate-House Problem for 1838. (*Cambridge Mathematical Journal*, Vol. I., p. 168, quoted by Salmon, *Conics*, page 99, 4th Ed.; see also *Reprint*, Vol. VIII., p. 48, Quest. 2444). In the more general form, we have the following:—If any transversal cuts the sides BC, CA, AB of a triangle in a, b, c respectively, the circles circumscribing ABC, Abc, Bca, Cab intersect in a point, and conversely. If D be the common intersection, (DA, DB, DC), (DA, Db, Dc), (Da, DB, Dc), (Da, Db, DC) cut off similar segments (*Quarterly Journal of Mathematics*, Vol. IV., p. 362).

These properties are projective, and as the circular form somewhat disguises the symmetry, we subjoin the result of a more general view. Let two conics Q$abcd$, R$abcd$ be drawn. Take fixed points S, T on these respectively, and suppose that the sides of the triangle PQR pass successively through S, a, T, while the vertices Q, R move on Q$abcd$, R$abcd$ respectively. Then, observing four positions of P and the corresponding anharmonic ratio, it is clear that the locus of P is a conic passing through STbcd. Consequently, with reference to any triangle PQR, if we take points S on PQ, a on QR, T on RP, and draw through PST, QaS, RaT, respectively, three conics intersecting in the same two points elsewhere, they will have a third common intersection. To circles having similar segments upon three several chords, correspond conics with respect to which the points Pbcd, Qbcd, Rbcd subtend equal anharmonic ratios when taken in corresponding order.]

2559. (Proposed by the Rev. R. TOWNSEND, F.R.S.)—Four quadrics pass each through three of four arbitrary lines in space; show that the four poles with respect to them of any arbitrary plane are coplanar.

Solution by W. S. M'CAY, B.A.

The two lines in space which intersect the four arbitrary lines being common generators of the four quadrics, and the two planes determined by them with the line connecting their two points of intersection with the arbitrary plane being consequently tangent planes to the four surfaces, therefore the fourth plane, harmonically conjugate to the arbitrary with respect to the two latter planes, contains the four poles in question, and therefore, &c.

2389. (Proposed by Professor CREMONA.)—Deux droites qui divisent harmoniquement les trois diagonales d'un quadrilatère rencontrent en quatre points harmoniques toute conique inscrite dans le quadrilatère.

Solution by T. COTTERILL, M.A.; REV. R. TOWNSEND, F.R.S.; *and others.*

Let Z be the intersection of two conjugate chords AA′, BB′ of a conic; then X, the pole of AA′, is on BB′; XA, XA′ are tangents, and XZ is harmonic to BB′. The anharmonic ratio of any point on the conic to the points AA′BB′ being the same as that of A (AA′BB′) or (XZBB′), is consequently harmonic. But in the quadrilateral, if two lines divide harmonically the three diagonals, they are conjugates to all the inscribed conics, and consequently, from what has been said, cut each conic in four points harmonic to the points upon it.

2498. (Proposed by R. BALL, M.A.)—Let S_n represent the sum of the nth powers of the roots of the cubic $ax^3+3bx^2+3cx+d=0$, and S'_n the result of interchanging a and d, b and c, in S_n; then prove that

$$a^{2n}S_{2n}=(a^nS_n)^2+2(-1)^{n+1}a^nd^nS'_n,$$

and hence show that

$$\begin{aligned}a^{12}S_{12}=&(3a^4d^2-108a^3bcd-54a^3c^3+729a^2b^2c^2+162a^2b^3d-1458ab^4c+729b^6)^2\\&-2a^6(3d^4a^2-108d^3cba-54d^3b^3+729b^2c^2d^2+162d^2c^3a-1458dc^4b+729c^6).\end{aligned}$$

Solution by SAMUEL ROBERTS, M.A.

If we represent by Π_1 the operator $3b\dfrac{d}{da}+2c\dfrac{d}{db}+d\dfrac{d}{dc}$, and by Π_2 the operator $3c\dfrac{d}{dd} + 2b\dfrac{d}{dc} + a\dfrac{d}{db}$, the sum of the nth powers of roots of the given equation will be given by $(-)^n \dfrac{\Pi_1{}^n}{[n-1]}\log(a)$, and the sum of the inverse nth powers will be given by $(-)^n \dfrac{\Pi_2{}^n}{[n-1]}\log(d)$. But

$$\alpha^{2n}+\beta^{2n}+\gamma^{2n} = (\alpha^n+\beta^n+\gamma^n)^2 - 2\alpha^n\beta^n\gamma^n\left(\frac{1}{\alpha^n}+\frac{1}{\beta^n}+\frac{1}{\gamma^n}\right),$$

and therefore we have the first result in the question.

As a particular case $a^{12}S_{12} = a^{12}\left(\dfrac{\Pi_1{}^6}{[5]}\log a\right)^2 - 2a^6d^6\left(\dfrac{\Pi_2{}^6}{[5]}\log d\right)$;

and performing the operations indicated, we get the second result. We may either avail ourselves of Arbogast's rules of abbreviation, or apply the general formula by making $n=3$.

2420. (Proposed by W. S. B. Woolhouse, F.R.A.S.)—A set of three numbers, each of them not exceeding a number n, being arbitrarily taken from a list of such combinations, let p_n denote the probability that they shall form the sides of a possible triangle; and show that, provided n be an even number, the consecutive probabilities p_n, p_{n+1}, p_{n+2} are in arithmetical progression.

Solution by the Proposer.

1. Assuming that repetitions are allowed, the total number of combinations contained in the list from which the set of numbers is taken is $\dfrac{n(n+1)(n+2)}{2.3}$; and if u denote the number of cases which exhibit the sides of a possible triangle, the sought probability will be

$$p = \frac{2.3u}{n(n+1)(n+2)}.$$

To determine the form of u, let n be supposed to become $n+1$ and the number of new favourable cases will express the value of the finite difference Δu. These combinations will all of them necessarily contain the new number $n+1$, which may be conveniently placed first as the highest in the order of magnitude. To obtain a possible triangle, in every case the sum of the other two numbers which follow must exceed the first number $n+1$. Hence, giving higher numbers always the precedence, if the second number of the triad be $n+1$ it may be associated with any one of the consecutive numbers from $n+1$ down to 1 inclusive. Also if the second number of a triad be n, it may be associated with numbers from n down to 2; if it be $n-1$ it may be taken with numbers from $n-1$ down to three inclusive, &c.

The numbers of cases thus presented are respectively $n+1$, $n-1$, $n-3$, &c.; and collecting them we get

$$\Delta u = (n+1)+(n-1)+(n-3)+(n-5)+\&c,$$

including all positive terms.

By reversing the order of the terms, we get

$$n \text{ even},\quad \Delta u = 1+3+5\ldots\ldots+(n+1) = \tfrac{1}{4}(n^2+4n+4),$$
$$n \text{ odd},\quad \Delta u = 2+4+6\ldots\ldots+(n+1) = \tfrac{1}{4}(n^2+4n+3).$$

Therefore, generally, $\Delta u = \frac{1}{8}\left\{2n^2+8n+7+(-1)^n\right\}$,

and by integration, making u vanish with x, we obtain

$$u_n = \frac{4n^3+18n^2+20n+3-3(-1)^n}{48}, \text{ and } p = \frac{4n^3+18n^2+20n+3-3(-1)^n}{8n(n+1)(n+2)}.$$

By separating the cases of n odd and even the expression becomes simplified in each. Thus

$$n \text{ even},\quad p = \frac{4n^3+18n^2+20n}{8n(n+1)(n+2)} = \frac{2n+5}{4(n+1)}$$
$$n \text{ odd},\quad p = \frac{4n^3+18n^2+20n+6}{8n(n+1)(n+2)} = \frac{(2n+1)(n+3)}{4n(n+2)}.$$

Hence if, according to the question, n be an even number we shall have

$$p_n = \frac{2n+5}{4(n+1)},\quad p_{n+1} = \frac{(2n+3)(n+4)}{4(n+1)(n+3)},\quad p_{n+2} = \frac{2n+9}{4(n+3)},$$

which are in a decreasing arithmetical progression, the common difference, or decrement, being $\dfrac{3}{4(n+1)(n+3)}$.

2. If repetitions are not admissible, the total number of combinations in the list from which the set of numbers is taken will then be $\dfrac{n(n-1)(n-2)}{2.3}$.

In this case, the new number $n+1$ being again placed first, if the second number of the triad be n, the third may be any number from $n-1$ down to 2; if the second number be $n-1$, it may be associated with numbers from $n-2$ down to 3, &c. The numbers of cases thus arising are severally $n-2$, $n-4$, &c.

Hence, collecting them, we have

$$n \text{ even},\quad \Delta u' = 2+4+6\ldots+n-2 = \tfrac{1}{4}(n^2-2n),$$
$$n \text{ odd},\quad \Delta u' = 1+3+5\ldots+n-2 = \tfrac{1}{4}(n^2-2n+1).$$

Therefore, generally, $\quad \Delta u' = \frac{1}{8}\left\{2n^2-4n+1-(-1)^n\right\}$,

$$u'_n = \frac{4n^3-18n^2+20n-3+3(-1)^n}{48}, \text{ and } p' = \frac{4n^3-18n^2+20n-3+3(-1)^n}{8n(n-1)(n-2)}.$$

This expression gives,

$$\text{for } n \text{ even},\quad p' = \frac{4n^3-18n^2+20n}{8n(n-1)(n-2)} = \frac{2n-5}{4(n-1)},$$
$$\text{for } n \text{ odd},\quad p' = \frac{4n^3-18n^2+20n-6}{8n(n-1)(n-2)} = \frac{(2n-1)(n-3)}{4n(n-2)}.$$

Hence, if n be even, we have

$$p'_n = \frac{2n-5}{4(n-1)},\quad p'_{n-1} = \frac{(2n-3)(n-4)}{4(n-1)(n-3)},\quad p'_{n-2} = \frac{2n-9}{4(n-3)},$$

which are likewise in arithmetical progression, the common difference being

$$\frac{3}{4(n-1)(n-3)}.$$

Note.—The values of the probability p in the two cases are mutually interchangeable by employing $-n$ instead of n.

It is further remarkable that

$$u_n + u_{n-1} = \frac{n(n+1)(n+2)}{2.3},\quad u'_n + u'_{n+1} = \frac{n(n-1)(n-2)}{2.3},$$

which determine the formulæ

$$p_n = 1 - \frac{n-1}{n+2}p_{n-1},\quad p'_n = \frac{n-3}{n}(1-p'_{n-1}).$$

II. *Solution by* STEPHEN WATSON.

If in the series, n being even, $n, n-1, n-2, \ldots .4, 3, 2, 1 \ldots\ldots\ldots (1)$, we take n as one number; $n-1$, $n-2$, successively, as another; and then in each case as many of the others as will form a triangle; we get the numbers of favourable cases as in the first line below. Next take $n-1$; and successively $n-2$, $n-3$: then in the same way we have the second line below; proceeding thus, we have

$$\begin{array}{l} n-3 \ldots\ldots +5+3+1 \\ n-4 \ldots\ldots +4+2 \\ n-5 \ldots\ldots +3+1 \\ n-6 \ldots\ldots +2 \\ \vdots \quad \ldots\ldots +1 \\ 2 \\ 1 \end{array}$$

Hence, adding vertically, we have the sum

$$= \tfrac{1}{2}(n-2)(n-3) \ldots +3.5+2.3+1 = \tfrac{1}{24}n(n-2)(2n-5).$$

In like manner, when n is odd, the number of possible triangles is

$$\tfrac{1}{24}(n-1)(n-3)(2n-1).$$

Also the total number of ways in which three numbers can be taken is $\frac{1}{6}n(n-1)(n-2)$; hence, according as n is even or odd,

$$p_n = \frac{2n-5}{4(n-1)},\qquad \therefore\ p_{n+2} = \frac{2n-1}{4(n+1)};$$

$$\text{or}\quad p_n = \frac{(n-3)(2n-1)}{4n(n-2)},\qquad \therefore\ p_{n+1} = \frac{(n-2)(2n+1)}{4(n^2-1)}\ \text{(here } n \text{ is even)}.$$

Therefore, when n is even, p_n, p_{n+1}, p_{n+2} are in arithmetical progression, for

$$p_{n+1} - p_n = p_{n+2} - p_{n+1} = \frac{3}{4(n^2-1)}.$$

[In solution the three numbers taken at random are supposed to be all different.]

2549. (Proposed by J. Griffiths, M.A.)—Solve the simultaneous equations $yz+x=14,\ zx+y=11,\ zy+z=10.$

Solutions by R. Tucker, M.A.; S. Bills; *and many others.*

1. Let $P^3=aP^2+bP-c=0$ be the equation whose roots are x, y, z; then

$$a+b=35 \ldots\ldots\ldots\ldots \text{(i.)}$$
$$b-3c+ab+ac=404 \ldots\ldots\ldots\ldots \text{(ii.)}$$
$$c+b^2-2ac+a^2c-2bc+c^2=1540 \ldots\ldots\ldots\ldots \text{(iii.)}$$

These equations reduce to

$$c^2+(a^2-69)\,c+a^2-70a-315=0 \ldots\ldots\ldots\ldots (\alpha),$$
$$(3-a)\,c+a^2-34a+369=0 \ldots\ldots\ldots\ldots (\beta);$$

whence we obtain

$$a^5-35a^4+258a^3+3454a^2-56331a+209709=0,$$

or, $$(a-9)\,(a^4-26a^3+24a^2+3670a+23301)=0.$$

Hence one set of answers is found from

$$a=9,\ b=26,\ c=24,$$

and $$P^3-9P^2+26P-24=(P-2)(P-3)(P-4)=0;$$

therefore $$x=2,\ y=3,\ z=4.$$

2. *Otherwise:* from the 2nd and 3rd equations we have

$$y=\frac{10x-11}{x^2-1},\quad z=\frac{11x-10}{x^2-1};$$

substituting these values in the 1st equation, it becomes

$$x^5-14x^4-2x^3+138x^2-220x+96=0,$$

or $$(x-2)\,(x^4-12x^3-26x^2+86x-48)=0;$$

whence $x=2$, and therefore $y=3$ and $z=4$.

2362. (Proposed by R. Tucker, M.A.)—Find the curves whose circles of curvature intercept upon a given line portions which (1) vary as the radii of curvature, (2) are constant.

Solution by the Proposer.

1. Taking the fixed line as axis of x, we have in the *first case* (since the intercept varies as the radius) the angle at the centre constant, and therefore β (the ordinate of the centre of curvature) varies as ρ.

$$\text{Let}\quad c\rho=\beta,\ \text{or}\quad \frac{q}{(1+p^2)\left\{c(1+p^2)^{\frac{1}{2}}-1\right\}}=\frac{1}{y},$$

$$\text{therefore}\quad \left\{\frac{c\,(1+p^2)^{\frac{1}{2}}+1}{1+p^2}-\frac{c^2}{c\,(1+p^2)^{\frac{1}{2}}-1}\right\}qp=-\frac{p}{y},$$

whence we get

$$c\,(1+p^2)^{\frac{1}{2}}+\log\,(1+p^2)^{\frac{1}{2}}+\log\left(\frac{y}{a}\right)=c\,(1+p^2)+\log\left\{c\,(1+p^2)^{\frac{1}{2}}-1\right\}$$

or $(c-y)(1+p^2)^{\frac{1}{2}} = a$, which gives the circle $(x+b)^2+(y-c)^2 = a^2$, where a, b, c are constants.

2. In the *second case*, we obtain

$$\rho^2-\beta^2 = k^2$$

if $2k$ is the constant intercept; which leads to the differential equation

$$q^2(k^2+y^2)+2yq(1+p^2) = p^2(1+p^2)^2,$$

of which the circle is a solution, as may be seen otherwise.

No other curves apparently can be found fulfilling the conditions.

1920. (Proposed by C. W. MERRIFIELD, F.R.S.)—What is the condition that an equation should have a pair of roots equal in magnitude and opposite in sign?

I. *Solution by* F. D. THOMSON, M.A.; S. BILLS; W. H. LAVERTY; *and others.*

Let $F(x) \equiv x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots\ldots+p_n = 0 \ldots\ldots\ldots\ldots\ldots (1)$,

be the proposed equation, of which a, $-a$ are roots; then

$$F(a) = 0,\ F(-a) = 0;$$

therefore $F(a)+F(-a) = 0 \ldots\ldots (2)$, $F(a) - F(-a) = 0 \ldots\ldots (3)$.

First let n be even and $= 2m$; then from (2) and (3), writing α for a^2,

$$\alpha^m+p_2\alpha^{m-1}+p_4\alpha^{m-2}+\ldots\ldots+p_n = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

$$p_1\alpha^{m-1}+p_3\alpha^{m-2}+\ldots\ldots+p_{n-1} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5);$$

and the result of eliminating α between the equations (4) and (5) will be the condition required.

If n be odd and $= 2m+1$, we obtain in like manner

$$\alpha^m+p_2\alpha^{m-1}+\ldots\ldots+p_{n-1} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6),$$

$$p_1\alpha^m+p_3\alpha^{m-1}+\ldots\ldots+p_n = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7);$$

and we have to eliminate α between (6) and (7).

II. *Solution by the* PROPOSER.

Let $\pm a$ be the pair of roots; then the equation must have x^2-a^2 as a factor. Now if we multiply this quantity (x^2-a^2) by an expression of the form

$$q_0x^{n-2}+q_1x^{n-3}+\ldots\ldots\ldots\ldots+q_{n-2} \ldots\ldots\ldots\ldots (1),$$

it is clear that in the resulting expression

$$p_0x^n+p_1x^{n-1}+\ldots\ldots\ldots\ldots+p_n \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

we shall get the odd terms (whose coefficients are of the form p_{2m}) from the odd terms (q_{2m-2}) of the expression (1) without regard to the even terms of either expression; and, similarly, that we shall get the even terms of (1) from the even terms of (2), independently of the odd terms.

The condition therefore is, that

$$p_0x^n+p_2x^{n-2}+\ldots\ldots, \text{ and } p_1x^{n-1}+p_3x^{n-3}+\ldots\ldots$$

should have a common measure.

III. *Solution by the* REV. J. L. KITCHIN, M.A.

1. Let $f(x)=0=(x-\alpha_1)(x+\alpha_1)(x-\alpha_3)\ldots\ldots\ldots(x-\alpha_n)$

therefore $f(-x)=(-)^n(x-\alpha_1)(x+\alpha_1)(x+\alpha_3)\ \ldots.\ (x+\alpha_n)$,

therefore $f(x)=0, f(-x)=0$, have a common measure $x^2-\alpha_1^2$.

Hence the most practicable condition for the roots in question is that found by writing $-x$ in the equation, and then seeking their Greatest Common Measure. The Greatest Common Measure will, if the remaining roots are unequal, be the quadratic factor containing the roots in question.

2. Let $x^n+p_1x^{n-1}+p_2x^{n-2}+\ \ldots\ldots+p_{n-2}x^2+p_{n-1}x+p_n=0\ldots\ldots(1)$; assume $x^2-\alpha_1^2$ as the factor which put equal to 0 gives the roots equal but of opposite sign, and

$$x^{n-2}+q_1x^{n-3}+q_2x^{n-4}+\ldots\ldots+q_{n-4}x^2+q_{n-3}x+q_{n-2}=0$$

the quotient of (1) by $x^2-\alpha_1^2=0$. From these last, we get

$$x^n+q_1x^{n-1}+(q_2-\alpha_1^2)\,x^{n-2}+(q_3-\alpha_1^2q_1)\,x^{n-3}+(q_4-\alpha_1^2q_2)\,x^{n-4}+\ \ldots\ldots$$
$$+(q_{n-2}-\alpha_1^2q_{n-4})\,x^2-q_{n-3}\alpha_1^2\,x-\alpha_1^2q_{n-2}=0.$$

Whence comparing with (1), we get

$$p_1=q_1,\ p_2=q_2-\alpha_1^2,\ p_3=q_3-\alpha_1^2q_1,\ p_4=q_4-\alpha_1^2q_2\ldots\ldots\ \ldots$$
$$p_{n-2}=q_{n-2}-\alpha_1^2q_{n-4},\ \ p_{n-1}=-\alpha_1^2q_{n-3},\ \ p_n=-\alpha_1^2q_{n-2};$$

$$\text{therefore } p_2-q_2=\frac{p_3-q_3}{p_1}=\frac{p_4-q_4}{q_2}=\frac{p_{n-2}-q_{n-2}}{q_{n-4}}=\frac{p_{n-1}}{q_{n-3}}=\frac{p_n}{q_{n-2}}.$$

Here there are $n-2$ equations, and $n-3$ unknown quantities, so their elimination will tend to an equation in $p_1, p_2\ldots.p_n$, which is the condition sought.

If the equation be a cubic, we get at once $p_1p_2=p_3$ as the condition.

In the same manner we find for a biquadratic the condition,

$$p_1p_2p_3=p_1^2p_4+p_3^2, \text{ and so on.}$$

As the equation increases in degree the condition becomes more and more complicated; in fact the simple device of changing x into $-x$ and seeking the Greatest Common Measure of the original and resulting equations is infinitely preferable as a means of discovering whether a given equation contains two roots equal, but opposite in sign, with the single exception of the cubic.

2367. (Proposed by F. A. TARLETON, M.A.)—If $x^n + b_1x^{n-1} + b_2x^{n-2} + \dots + b_n = 0$ be an algebraical equation whose roots are $x_1, x_2, \dots x_n$ show that

$$\begin{vmatrix} \frac{db_1}{dx_1}, & \frac{db_2}{dx_1} & \frac{db_3}{dx_1} & \dots & \frac{db_n}{dx_1} \\ \frac{db_1}{dx_2}, & \frac{db_2}{dx_2}, & \frac{db_3}{dx_2} & \dots & \frac{db_n}{dx_2} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \frac{db_1}{dx_n} & \dots\dots & \dots\dots & \dots\dots & \frac{db_n}{dx_n} \end{vmatrix} = \sqrt{(\Delta)},$$ where Δ is the discriminant.

Solution by J. DALE; *and* R. W. SYMES, B.A.

Neglecting the sign for the present, we have

$$b_1 = \Sigma x_1, \; b_2 = \Sigma x_1x_2, \dots b_n = \Sigma x_1x_2 \dots x_n;$$

$$\frac{db_1}{dx_1} = \frac{db_1}{dx_2} = \frac{db_1}{dx_3} = \dots\dots\dots\dots = \frac{db_1}{dx_n} = 1;$$

$$\frac{db_2}{dx_1} = b_1 - x_1, \; \frac{db_2}{dx_2} = b_1 - x_2, \dots\dots \frac{db_2}{dx_n} = b_1 - x_n;$$

$$\frac{db_3}{dx_1} = b_2 - b_1x_1 + x_1^2; \; \frac{db_3}{dx_2} = b_2 - b_1x_2 + x_2^2; \; \dots\dots$$

$$\frac{db_m}{dx_n} = b_{m-1} - b_{m-2}x_n + \dots\dots\dots \pm (b_1x_n^{m-2} - x_n^{m-1}).$$

Substituting these values in the determinant, it becomes, after reduction,

$$\begin{vmatrix} 1 & x_1 & x_1^2 & x_1^3 \dots\dots x_1^{n-1} \\ 1 & x_2 & x_2^2 & x_2^3 \dots\dots x_2^{n-1} \\ 1 & x_3 & x_3^2 & x_3^3 \dots\dots x_3^{n-1} \\ \vdots & \vdots & \vdots & \vdots \; \text{- - - -} \; \vdots \\ 1 & x_n & x_n^2 & x_n^3 \dots\dots x_n^{n-1} \end{vmatrix}$$

Subtracting the last line from each of the others, dividing out, and repeating the operation, the determinant becomes

$$(x_1 - x_2) \dots (x_1 - x_n)\,(x_2 - x_3) \dots (x_2 - x_n) \dots (x_{n-1} - x_n);$$

therefore the square of the determinant is equal to the last or absolute term in the equation to the squares of the differences of the given equation, that is, to the discriminant of the given equation. Hence the theorem is proved.

[The method used by Dr. SALMON (*Higher Algebra*, p. 13, ex. 5) for finding the value of the determinant to which Mr. DALE reduces the question is immediately applicable to the determinant in the question itself. Thus we see at once that, if two roots are equal, the determinant vanishes; hence $(x_1-x_2)\dots(x_{n-1}-x_n)$ is a factor. Also, since a root appears only in the first degree, and in $n-1$ of the rows, the order in which a root appears in the worked out result is $n-1$, by a known rule. Hence there can only be a numerical factor besides, and this is proved to be unity by observing that if $x_n = x_{n-1} = \dots = x_3 = 0$, the numerical multiplier is not altered; thus, if M be the multiplier, we have $Mx_1x_2(x_1-x_2) = x_1x_2(x_1-x_2)$.]

1934. (Proposed by M. W. Crofton, B.A.) — Two bags contain m and n balls respectively; an arbitrary number is drawn from each, 0 being considered a number [and all numbers being supposed equally probable]: find the chance of the total number drawn being equal to any assigned integer, from 0 to $m+n$.

Solution by Stephen Watson.

Let m be $>$ n, and r the assigned number, and take the question in the following cases:—

1. When r is not $>$ n. In this case the number of ways favourable to the chance is the number of ways in which any two of the numbers 0, 1, 2, ..., each figure being taken twice, can be added so as to give r, and is $r+1$.

2. When r $>$ n but not $>$ m. Here the favourable cases are $n+1$.

3. When $r>m$. The favourable cases are $n-\{r-(m+1)\}$. Hence, $(m+1)(n+1)$ being the total cases, the chances in the three cases are

$$\frac{r+1}{(m+1)(n+1)},\quad \frac{1}{m+1},\quad \frac{n-\{r-(m+1)\}}{(m+1)(n+1)}.$$

[If we denote these functions by F_1, F_2, F_3, respectively, we have

$$F_1(m, n, r) + F_3(m, n, r) = F_2(m, n, r) + F_2(n, m, r).$$

See *Reprint*, Vol. IV., p. 86.]

From these the respective chances, of some number not greater than n, of some number $>$ n but not $>$ m, and of some number $>$ m but not $>$ $m+n$, are easily found to be

$$\frac{\frac{1}{2}n+1}{m+1},\quad \frac{m-n}{m+1},\quad \frac{\frac{1}{2}n}{m+1};$$

and the sum of these is unity, as it ought to be.

[In the above solution, to which the amended enunciation of the question is adapted, it is supposed that all possible numbers have severally an equal chance of being drawn from each bag. The solution may also be regarded as a strictly accurate one to the following question:—

Two bags contain $m+1$ and $n+1$ counters, numbered 0, 1, 2 ... m, and 0, 1, 2 ... n, respectively; a counter is drawn from each; find the chance of the total number drawn being equal to any assigned integer, from 0 to $m+n$.

If we take the problem in its general and unrestricted form, and suppose that balls are drawn at random from each bag, that is to say, that all the possible ways in which a ball or balls can be drawn are equally probable, the chance of drawing any given number r in the manner stated in the question will evidently be unchanged if a single drawing be made from one bag containing all the $m+n$ balls; and its value will therefore be the coefficient of x^r in $\left(\frac{1+x}{2}\right)^{m+n}$.]

NOTE ON QUESTION 2471 : BY W. S. B. WOOLHOUSE, F.R.A.S.

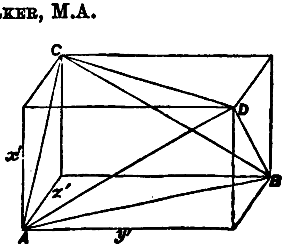
According to one of the formulæ stated in Question 2471 [*Reprint*, Vol. VIII., p. 103], the average square of the area of a triangle, formed by joining three arbitrary points, is $= \frac{3}{2} h^2 k^2$. This simple and curious relation applies equally when the three points are taken on any curve line, or on any kind of line whatever, if h, k still denote the radii of gyration of the same with respect to the principal axes passing through the centre of gravity.

It may also be observed that in the general question, as proposed, the formulæ which express the values of (Δ) and (Δ^2) are free from any condition as to convexity of boundary. For both of these formulæ the form of boundary is absolutely unrestricted. But the subsistence of the subsequent formulæ, which express the values of the several probabilites, requires the condition of a convex boundary, since in the ulterior reasoning upon which they are founded in the solution of the question, it is tacitly implied that the several straight lines which connect the points lie wholly upon the given surface or within the given boundary.

[The substance of this note was intended to be annexed to the solution, but was accidentally omitted.]

1496. (Proposed by M. COLLINS, B.A.)—Prove that a triangular pyramid whose vertices are ABCD, and a parallelepiped formed from it as follows, have the same centre of gravity; viz. through any point in each of the opposite edges AB and CD draw straight lines parallel to the other edge, we thus get two parallel plane faces of the parallelepiped; two other parallel faces of it are similarly obtained from the opposite edges AC and DB; and the third pair of faces are obtained from the remaining two opposite edges AD and BC.

Solution by J. J. WALKER, M.A.



The theorem proposed for proof in this question coincides with one to which I was led in considering "some analogues in space to properties of the parallelogram" (see *Messenger of Mathematics*, Vol. IV., p. 144), inasmuch as the construction directed gives a parallelepiped related to the tetrahedron thus: the vertices BCD are the corners opposite to A of the three parallelogram-faces of the parallelepiped which cointersect at A. Call the three edges of the parallelepiped (x', y', z'). Then the equation to the plane BCD referred to the three edges cointersecting in A will be $xy'z'+yx'z'+zx'z'-2x'y'z'=0$, since this equation is evidently satisfied by the coordinates $(0, y', z')$, $(x', 0, z')$, $(x', y', 0)$ of BCD. Also the plane passing through the centre of gravity of the tetrahedron, and parallel to BCD, will have as its equation

$$2xy'z'+2yz'x'+2zx'y'-3x'y'z'=0,$$

(since this plane intercepts on the axes lengths which are $\frac{3}{4}$ of those intercepted by the plane BCD,) and will evidently pass through the point $(\frac{1}{2}x'\ \frac{1}{2}y', \frac{1}{2}z')$, that is, through the centre of gravity of the parallelepiped. Similarly planes drawn through the centre of gravity of the tetrahedron, and parallel to the other three faces, may be shown to pass through that of the parallelepiped. Hence these two points coincide.

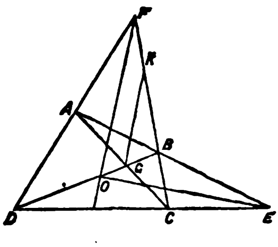
[The centre of gravity of the pyramid is the centre of gravity of four equal particles at its corners, and therefore bisects the straight line joining the middle points of AB, CD; it coincides, therefore, with the centre of gravity of the parallelepiped. Another solution is given on p. 87 of Vol. VIII. of the *Reprint*.]

2411. (Proposed by Professor SYLVESTER.)—

1. Prove geometrically that the quadrangle formed by the intersections of one pair of right lines equally inclined at any angle to a given direction with another pair equally inclined at any other angle thereto, is inscribable in a circle, and that the third pair of sides of such quadrangle will also be equally inclined to the same direction.

2. Hence obtain, as an instantaneous algebraical inference, the theorem that the intersection of any two right cones with parallel axes is a spherical curve contained in a known sphere and a known third right cone whose axis is parallel to those of the given two.

Solution (1) *by the* REV. J. WOLSTENHOLME, M.A.; (2) *by the* PROPOSER.



1. Let the quadrangle be ABCD; and let FO, EO be straight lines bisecting the angles E, F, and by the conditions of the question at right angles to each other. Then we have

$$\angle \text{ABC} = \text{BFO}+\text{FOE}+\text{OEB}$$
$$= \tfrac{1}{2}\text{F}+\text{O}+\tfrac{1}{2}\text{E},$$

and $\angle \text{ADC} = \text{FOE}-\text{OED}-\text{DFO}$
$$= \text{O}-\tfrac{1}{2}\text{F}-\tfrac{1}{2}\text{E},$$

$\therefore\ \angle \text{ABC}+\angle \text{ADC} = \text{twice O}$
(this is general) $= \text{two right angles}$;
or A, B, C, D lie on a circle.

If AC, BD meet in G, and GK bisecting the angle ACB meet FB in K,

$$2\angle \text{GKB} = 2\text{DBC}-\text{AGB} = \text{DBC}-\text{ACB} = \text{DAC}-\text{ACB} = \text{DFC},$$

therefore $\angle \text{GKB} = \tfrac{1}{2}\text{DFC} = \text{OFC}$; or GK is parallel to OF.

[Mr. Wolstenholme remarks that he set this problem in the year 1859 at Christ's College, Cambridge. See also the solution of Quest. 1950, *Reprint*, Vol. V., p. 105.]

2. We now see that if

$$\text{F} = (x-a)^2-m(z-c)^2, \text{ and } \text{G} = (x-a)^2-m'(z-c')^2,$$

there must exist a ratio $p : q$ such that

$$\frac{pF+qG}{p+q} = H = (x-a'')^2 - m''(z-c'')^2,$$

where the three pairs of lines F, G, H will intersect in the angles of a quadrangle inscribable in a circle, and that accordingly another ratio $r : s$ may be assumed such that

$$\frac{rF+sG}{r+s} = K = (x-\alpha)^2 + (z-\gamma)^2 = C.$$

Hence it follows that

$$\frac{p(F+y^2)+q(G+y^2)}{p+q} = H+y^2, \text{ and } \frac{r(F+y^2)+s(G+y^2)}{r+s} = K+y^2;$$

so that, if $(x-a)^2+y^2 = m(z-c)^2$...(1) and $(x-a')^2+y^2 = m'(z-c')^2$...(2),

we must also have $(x-\alpha)^2+y^2+(z-\gamma)^2 = C$ ....(3),

and $(x-a'')^2+y^2 = m''(z-c'')^2$ ....(4),

which proves the theorem, and shows how the containing third cone and sphere may be geometrically determined.

The mere existence of a containing sphere and third cone, or several such (real or imaginary), might of course also be inferred *à priori* from an enumeration of the disposable constants by aid of which (1) and (2) admit of being brought into linear equivalence with (3) and with (4). There will in each case be four such disposable constants $\frac{p}{q}$, α, γ, C; or $\frac{p}{q}$, a'', c'', m''; to satisfy four equations arising out of the coefficients of z^2, z, x, 1.

2573. (Proposed by Professor CAYLEY.)—The envelope of a variable circle having for its diameter the double ordinate of a rectangular cubic is a Cartesian. [DEF.—The expression "a rectangular cubic" is used to express a cubic with three real asymptotes, having a diameter at right angles to one of the asymptotes and at an angle of 45° to each of the other two asymptotes, viz., the equation of such a cubic is $xy^2 = x^3 + bx^2 + cx + d$.]

Solution by the PROPOSER.

The equation of the variable circle may be taken to be

$$(x-\theta)^2 + y^2 = \theta^2 - 2m\theta + a + \frac{2A}{\theta},$$

viz., θ being the abscissa of the rectangular cubic, the squared ordinate is taken to be $= \frac{1}{\theta}(\theta^3 - 2m\theta^2 + a\theta + 2A)$, or, what is the same thing, the equation of the variable circle is $x^2+y^2-a-2(x-m)\theta - \frac{2A}{\theta} = 0$.

Hence, taking the derived equation in regard to θ, we have

$$x-m-\frac{A}{\theta^2}=0, \text{ and thence } x^2+y^2-a=\frac{4A}{\theta};$$

therefore $$(x^2+y^2-a)^2=\frac{16A^2}{\theta^2}=16A(x-m);$$

that is, the equation of the envelope is

$$(x^2+y^2-a)^2-16A(x-m)=0,$$

which is a known form of the equation of a Cartesian.

[Mr. WOLSTENHOLME obtains the equation of the envelope in the equivalent form

$$(x^2+y^2-c)^2=4d(b+2x)\dots\dots\dots\dots\dots\dots\dots(1),$$

and proceeds with his solution as follows:—

The axis of x divides (1) symmetrically, and if it be a Cartesian its foci will be in this axis: let two of these be at distances α, β from the origin, and let the equation of the Cartesian be

$$m\left\{(x-\alpha)^2+y^2\right\}^{\frac{1}{2}}+n\left\{(x-\beta)^2+y^2\right\}^{\frac{1}{2}}=1.$$

This, on developing and putting $m^2\alpha=n^2\beta$, becomes

$$\left\{x^2+y^2-\frac{m^2\alpha^2}{n^2}-\frac{m^2+n^2}{(m^2-n^2)^2}\right\}^2=\frac{-8m^2\alpha x}{(m^2-n^2)^2}+\frac{4m^2\alpha^2(m^2+n^2}{n^2(m^2-n^2)^2}+\frac{4m^2n^2}{(m^2-n^2)^4}\dots(2),$$

which coincides with (1) if

$$c=\frac{m^2\alpha^2}{n^2}+\frac{m^2+n^2}{(m^2-n^2)^2},\quad d=\frac{-m^2\alpha}{(m^2-n^2)^2},\quad -b=\frac{\alpha(m^2+n^2)}{n^2}+\frac{n^2}{\alpha(m^2-n^2)^2}.$$

These give immediately the equation for α, viz. $\alpha^3+b\alpha^2+c\alpha+d=0$, a cubic whose three roots give the positions of the three foci.]

2358. (Proposed by Professor CREMONA.)—La condition qu'une conique divise harmoniquement les trois diagonales d'un quadrilatère circonscrit à une autre conique, coincide avec la condition que la première conique soit circonscrite à un triangle conjugué à l'autre.

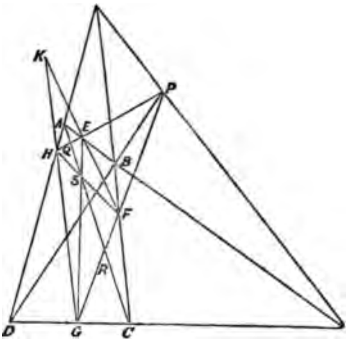
Solution by T. COTTERILL, M.A.

If a conic U cut the diagonals of a quadrilateral harmonically, the three pairs of lines through its intersections with two diagonals are evidently conjugates to a conic Σ inscribed in the quadrilateral. But if two lines are conjugate to a conic, the polar to the conic of any point on one of the lines cuts the lines in a pair of points harmonics to the points in which it meets the conic. Hence the polar to Σ of a point of intersection of U with one of the two diagonals cuts Σ in two points conjugates to its intersections with the three pairs of lines; and, by a well known theorem, the same points on

Σ are conjugates to the intersections of the line with U. But if a line cuts two conics U and Σ harmonically, and the pole of the line to Σ is on U, then this pole and the two points of intersection of the line and U form a triangle conjugate to Σ.

NOTE ON QUESTION 2473: BY THE REV. J. WOLSTENHOLME, M.A.

The distribution of ten points into ten rows of three in a row may be very simply done by the following construction:— Take any quadrangle ABCD, and from a point P on the intersection of two diagonals draw any two straight lines, meeting the third diagonal in Q, R, and the sides in E, F, G, H. Then (PEHQ) and (PGFR) are both harmonic ranges. Therefore FH and EG will meet in QR, in S suppose. Then the ten points A, B, C, D, E, F, G, H, P, S are ten points in the ten rows AHD, ASC, AEB, PBD, BFC, CGD, HEP, ESG, FSH, PFG.



If EF and GH intersect in K, then the above-mentioned ten points and K form eleven points in twelve rows of three.

[Professor SYLVESTER remarks that the general theorem in Quest. 2572 shows that with 11 points $\frac{1}{6}(10 \cdot 9)$, *i.e. fifteen*, rows of three may be formed, such points being pluperfect points of the 11th order lying on a cubic curve.]

2543. (Proposed by M. W. CROFTON, B.A.)—1. Given any closed convex boundary of length L, and another of length l lying wholly *inside* the former; the probability that a line drawn at random in the plane and meeting L shall also meet l, is l : L. (N.B.—If the boundaries are not convex, this expression still holds good, if L denote, not the length of the boundary, but that of a string tightly wrapped round it; likewise for l.)

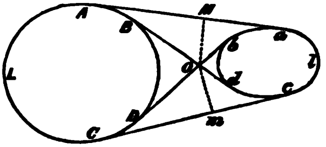
2. If l be the length of a closed convex boundary lying wholly *outside* L, X the length of an endless band tightly enveloping the two boundaries and crossing between them, and Y the length of another endless band also enveloping both but not crossing between them, the probability that a random line meeting L shall also meet l, is $(X-Y) : L$.

3. If the boundary l intersects L; then, if Y be the length of an endless band tightly enveloping both, the probability is $(L+l-Y) : L$.

Solution by CAPTAIN A. R. CLARKE, R.E., F.R.S.

1. Let O be any point within a closed convex curve, p, p' the perpendiculars on the tangents which make an angle θ with a given line; then $p+p$ is the breadth of the curve corresponding to the direction θ, and consequently $p+p'$ represents the number of random lines, parallel to θ, which cut the curve. Hence the total number of random lines cutting the curve is easily seen to be

$$\int_0^\pi (p+p')\,d\theta,$$ which is, by a known theorem (Todhunter's *Int. Calc.*, p. 88), equal to the perimeter of the curve. Hence, if there be a closed curve of length l within another of length L, the chance that a random line cutting the second also cuts the first is l : L.



2. Again, if l be without L; let $\theta=0$ correspond to Bd, θ_1 to Aa, θ_2 to Cc, θ_3 to Db. Let the perpendiculars on tangents be drawn from O, the intersection of the tangents Bd, Db. Let P$'$, p' be perpendiculars on tangents to the upper parts of L and l respectively; P$_{,}$, $p_{,}$ perpendiculars on tangents at the lower parts, then a little consideration will show that the number of random lines crossing both curves is

$$\text{(between 0 and } \theta_1) = \int_0^{\theta_1} (P'+p_{,})\,d\theta, \quad \text{(between } \theta_1 \text{ and } \theta_2) = \int_{\theta_1}^{\theta_2} (p'+p_{,})\,d\theta,$$

$$\text{(between } \theta_2 \text{ and } \theta_3) = \int_{\theta_2}^{\theta_3} (p'+P_{,})\,d\theta, \text{ which added are equal to}$$

$$\int_0^{\theta_1} P'\,d\theta \;(= \mathrm{AB+BO-AM}) + \int_{\theta_2}^{\theta_3} P_{,}\,d\theta \;(= \mathrm{CD+DO-C}m)$$

$$+ \int_{\theta_1}^{\theta_3} p'\,d\theta \;(= ab + \mathrm{O}b - a\mathrm{M}) + \int_0^{\theta_2} p_{,}\,d\theta \;(= cd + \mathrm{O}d - cm)$$

$$= \mathrm{AB+CD}+ab+cd+\mathrm{D}b+\mathrm{B}d-\mathrm{A}a-\mathrm{C}c,$$

and this is easily seen to be $=\mathrm{X-Y}$, so that the probability in this case is (X − Y) : L.

3. If the curves intersect in two points, let the perpendiculars be dropped from a point O within both curves; then, if θ_1 be the inclination of the tangents Aa, Cc, the number of random lines crossing both curves is

$$\int_0^{\theta_1} (p_{,}+p')\,d\theta + \int_{\theta_1}^{\pi} (P'+p_{,})\,d\theta = \int_0^{\pi} p_{,}\,d\theta + \int_0^{\theta_1} p'\,d\theta + \int_{\theta_1}^{\pi} P'\,d\theta,$$

which, combining $p_{,}$ with p', and integrating the sum from 0 to $\pi+\theta$, is seen to be equal to $ca - a\mathrm{M} - cm + \mathrm{CA} - \mathrm{C}m - \mathrm{AM} = \mathrm{CA} + ca - \mathrm{C}c - \mathrm{A}a$, to which if we add and subtract ALC + alc, we shall find the required number to be $= \mathrm{L}+l-\mathrm{Y}$. This expression will be found to hold good if the curves intersect in *four* points, in which case Y has four straight portions.

4. In a similar manner it may be shown that, if there be two non-intersecting convex closed curves, whose lengths are l, $l_{,}$ within a third whose

length is L, the chance that a random line intersecting L passes between l and l' is $(X-l-l') : L$.

NOTE.—If, as in the foregoing solution, we take a random line in accordance with the remarks of Mr. CROFTON on p. 85 of Vol. VII. of the *Reprint*, it may be similarly shown that, if two such random lines cross a closed convex curve whose length is L and area A, the chance that they intersect within the figure is $2\pi A : L^2$.

[Mr. CROFTON gives the following examples of the theorems, a straight line being viewed as an infinitely thin rectangle. A random straight line cuts a circle, the chance of its meeting an equal circle touching the first is $\dfrac{\pi-2}{\pi}$. The chance of a straight line, which cuts one side of a square, meeting the opposite side is $\sqrt{2}-1$; that of its meeting an adjacent side is $1-\frac{1}{2}\sqrt{2}$. The chance of a straight line, which meets a semicircle, meeting the opposite semicircle is $\dfrac{4}{\pi+2}$.]

2502. (Proposed by J. GRIFFITHS, M.A.)—If A, B, C, D be functions of x, y, z; and A_1, B_1, C_1, D_1 be certain constants; required the envelope of $A+\lambda B+\mu C+\nu D = 0$, when the variables λ, μ, ν are connected by the two equations $A_1+\lambda B_1+\mu C_1+\nu D_1 = 0$, $\lambda^2+\mu^2+\nu^2-2\lambda\mu\nu = 1$.

Solution by STEPHEN WATSON.

Eliminating first ν and then μ from the two first given equations, and substituting the resulting values of μ and ν in the third equation, we obtain

$$K\lambda^3+L\lambda^2+M\lambda+N = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where
$$K = 2(BC_1-B_1C)(DB_1-D_1B),$$

$$L = 2(AC_1-A_1C)(DB_1-D_1B) + 2(BC_1-B_1C)(DA_1-D_1A)$$
$$-(BC_1-B_1C)^2-(DB_1-D_1B)^2-(D_1C-DC_1)^2,$$

$$M = 2\left\{(AC_1-A_1C)(DA_1-D_1A)-(AC_1-A_1C)(BC_1-B_1C)\right.$$
$$\left.-(DA_1-D_1A)(DB_1-D_1B)\right\},$$

$$N = (D_1C-DC_1)^2-(AC_1-A_1C)^2-(DA_1-D_1A)^2.$$

Differentiating (1) with respect to λ, we have

$$3K\lambda^2+2L\lambda+M = 0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

and eliminating λ from (1) and (2),

$$(LM-9KN)^2 = 4(L^2-3KM)(M^2-3LN) \quad\ldots\ldots\ldots\ldots(3);$$

whence, substituting the above values of K, L, M, N, we have the equation of the envelope required.

2415. (Proposed by Dr. SALMON, F.R.S.)—Find the equation of the evolute to the Cissoid $x(x^2+y^2) = ay^2$.

I. *Solution by* Stephen Watson.

Since $y = \dfrac{x^{\frac{3}{2}}}{(a-x)^{\frac{1}{2}}}$, $\therefore\ \dfrac{dy}{dx} = p = \dfrac{x^{\frac{1}{2}}(3a-2x)}{2(a-x)^{\frac{3}{2}}}$, and $\dfrac{d^2y}{dx^2} = p'' = \dfrac{3a^2}{4x^{\frac{1}{2}}(a-x)^{\frac{5}{2}}}$; hence, (α, β) being a point in the required locus, we have

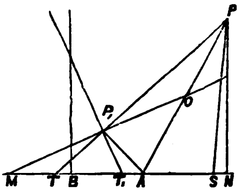
$$\alpha = x - \frac{p'(p'^2+1)}{p''} = \frac{ax(5x-6a)}{6(a-x)^2} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$\beta = y + \frac{p'^2+1}{p''} = \frac{4ax^{\frac{1}{2}}}{3(a-x)^{\frac{1}{2}}} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The elimination of x from (1) and (2) gives the required equation, which is, writing now (x, y) for (α, β),

$$27y^4 + 288a^2y^2 + 512a^3x = 0.$$

II. *Solution by* James Dale.



Let PT be the tangent at any point (h, k) of a parabola whose vertex is A, focus S, and parameter $4a$; then P_1, the foot of a perpendicular from A on PT, is a point on the cissoid

$$x(x^2+y^2) + ay^2 = 0.$$

If O be the middle point of AP, then OP_1 is a normal to the locus of P_1; that is, the evolute to the cissoid is the envelope of OP_1.

The equation to OP_1 is

$$(y - \tfrac{1}{2}k) = (x - \tfrac{1}{2}h)\tan \mathrm{OMA};$$

but $\quad \angle\mathrm{OMA} = \mathrm{APS} = \mathrm{APN} - \mathrm{SPN}$, therefore $\tan \mathrm{OMA} = \dfrac{ak}{h^2+k^2-ah}$.

Eliminating h by means of the equation $k^2 = 4ah$, the equation to OP_1 becomes $\qquad (12a^2k + k^3)y = 16a^3x + 4a^2k^2 + \frac{1}{2}k^4.$

Differentiating this with respect to k, and putting the result $= 0$, we get $2k = 3y$; and substituting in the equation to OP_1, and changing the sign of x, the equation to the evolute required is

$$y^2(9a^2 + \tfrac{27}{32}y^2) + 16a^3x = 0, \quad\text{or}\quad 27y^4 + 288a^2y^2 + 512a^3x = 0.$$

[A solution by Mr. Tucker is given on p. 58 of Vol. I. of the *Reprint*, where the form of the curve is shown. Mr. Tucker's result agrees with that given above, if we bear in mind that his a is the *radius* of the circular generator of the cissoid, whereas in the above solutions a is the *diameter* of the circle.]

2569. (Proposed by S. Roberts, M.A.)—If through a point O there be drawn three lines parallel respectively to the sides of a given triangle, the six points in which these lines meet the sides lie on a conic. Find its equation referred to the triangle. (See Question 2489.)

I. *Solution by* Stephen Watson.

Take the figure in the Solution to Quest. 2489 in the January number, use triangular coordinates, and denote O by (x', y', z'); then the coordinates of I and F are $(0,\ 1-z',\ z')$ and $(0,\ y',\ 1-y')$; hence the equations of AI and AF are
$$y = \frac{1-z'}{z'}z' \quad\text{and}\quad \frac{1-y'}{y'}y = z,$$
and therefore by symmetry the equation of the conic through the six points is
$$\left(y - \frac{1-z'}{z'}z\right)\left(\frac{1-y'}{y'}y - z\right) + \left(z - \frac{1-x'}{x'}x\right)\left(\frac{1-z'}{z'}z - x\right) +$$
$$\left(x - \frac{1-y'}{y'}y\right)\left(\frac{1-x'}{x'}x - y\right) - \frac{1-x'}{x'}x^2 - \frac{1-y'}{y'}y^2 - \frac{1-z'}{z'}z^2 = 0, \text{ or}$$
$$\frac{1-x'}{x'}x^2 + \frac{1-y'}{y'}y^2 + \frac{1-z'}{z'}z^2$$
$$- \left(2 + \frac{z'}{x'y'}\right)xy - \left(2 + \frac{x'}{y'z'}\right)yz - \left(2 + \frac{y'}{z'x'}\right)zx = 0.$$

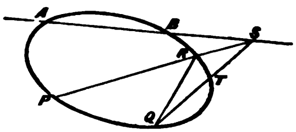
II. *Solution by the* Proposer.

Let (x, y, z) be the current trilinear coordinates referred to the given triangle. Then, denoting its sides by a, b, c, and writing L, M, N for $b\beta + c\gamma,\ a\alpha + c\gamma,\ a\alpha + b\beta$, where α, β, γ are the coordinates of the point O, we have for the coordinates of the six points of intersection
$$(0,\ \beta c,\ \text{M}),\quad (\alpha b,\ \text{L},\ 0),\quad (\text{N},\ 0,\ \gamma a),$$
$$(0,\ \text{N},\ \gamma b),\quad (\text{M},\ \beta a,\ 0),\quad (\alpha c,\ 0,\ \text{L});$$
and for conics through the first two pairs of points the equation is of the form
$$(\text{LM}x - \alpha b\text{M}y + \alpha\beta bcz)(\beta\gamma abx - \gamma b\text{M}y + \text{MN}z) + kxz = 0.$$
To determine k so that a conic of the system may pass through one of the remaining points, we have
$$k + \tfrac{1}{3}(\text{L}+\text{M}+\text{N})(\text{LMN} + abc\alpha\beta\gamma) = 0;$$
and substituting, we have
$$\frac{a\text{L}}{\alpha}x^2 + \frac{b\text{M}}{\beta}y^2 + \frac{c\text{N}}{\gamma}z^2 - \left(\frac{\text{LM}}{\alpha\beta} + ab\right)xy - \left(\frac{\text{MN}}{\beta\gamma} + bc\right)yz - \left(\frac{\text{LN}}{\alpha\gamma} + ac\right)zx = 0.$$
This may be put into a better form by adding $ax^2 + by^2 + cz^2$, and then subtracting the same. Thus we get
$$a(\beta\gamma x^2 - \alpha^2 yz) + b(\alpha\gamma y^2 - \beta^2 xz) + c(\alpha\beta z^2 - \gamma^2 xy) - \frac{\alpha\beta\gamma}{a\alpha + b\beta + c\gamma}(ax + by + cz)^2 = 0,$$
which is the required equation.

2554. (Proposed by Professor Hirst.)—Rays being drawn from any point on a conic; the segments intercepted thereon between the conic and a fixed

tangent can all be projected upon a given line, from another point on the conic, into segments of equal length.

I. *Solution by* W. S. McCay, B.A.; *the* Proposer; *and others.*



Draw any line through P meeting the conic and the line AB in R and S; and let Q be any other point on the conic. Then as the points R and T vary, RT divides the conic homographically, for clearly the anharmonic ratio of the four points R is that of the four points S, or that of the four points T. Also A and B are the two points which have the same conjugate considered as of either set. Now the relation between the distances of two conjugates of a homographic range on a line from a fixed point on the line is

$$axx' + bx + cx' + d = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1);$$

and the equation that gives the points that have the same conjugates in either system is $ax^2 + (b + c)\,x + d = 0$;

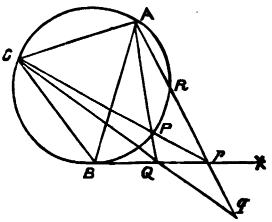
and if these are both infinite, then $a = 0,\ b + c = 0 \ldots\ldots\ldots\ldots\ldots (2)$,

and (1) reduces to $x - x' = -\dfrac{d}{b}$, that is, the distance between two conjugates is constant.

Now (2) will be the case, in the above figure, for a transversal across the homographic pencil whose vertex is Q, provided the line AB touch the conic, and the transversal be parallel to the line joining Q to the point of contact.

Hence the required construction for Q:—Draw a line through the point of contact of the tangent parallel to the direction of the given line: it meets the conic in Q.

II. *Solutions by* C. W. Merrifield, F.R.S.; H. M. Taylor, B.A.; J. J. Walker, M.A.; *and others.*



1. The theorem being a perspective one, the conic may be made a circle, and the given line may be drawn through the given point, without loss of generality.

Let ARBC be the given circle, BX the given tangent, AR*pq* the given line, and APQ a variable transversal.

Draw BC through the point of contact B, parallel to the given line, and cutting the conic in C; then C is evidently the point of sight required. For the segment of AB is infinitesimal, and its projection on AX can only be finite when the point of sight is on BC. As it is to be on the conic, it must be at B or C. It is evidently not at B, for the projection from B varies from zero to RX as Q moves from B to X. Hence it can only be at C, and it remains simply to show that the length *pq* in the figure is independent of the direction AQ.

Take BX, BC for Cartesian axes, and put $\angle\,\mathrm{CBX} = \omega$, $\mathrm{BC} = y_2$, $\mathrm{BQ} = x$; then the equation to the circle is

$$\mathrm{U} \equiv x^2 + y^2 + 2xy\cos\omega - yy_2 = 0.$$

Let A be (x_0, y_0): the equations to AQ and BC will be $y = y_0 \dfrac{x-x_1}{x_0-x_1}$ and $x=0$, and the quadratic equation of the pair of lines may be written as

$$V \equiv xy - \frac{y_0x^2}{x_0-x_1} + \frac{x_1y_0x}{x_0-x_1} = 0.$$

Again, BA is $y - \dfrac{y_0}{x_0}x = 0$; and if we write CP as $\lambda x = y - y_2$, where λ is indeterminate, we have

$$\text{BA}.\text{CP} \equiv V_1 \equiv y^2 - \left(\lambda + \frac{y_0}{x_0}\right)xy + \frac{\lambda y_0}{x_0}x^2 - yy_2 + \frac{y_0y_2}{x_0}x = 0.$$

Now U, V, V_1 have a common intersection in four points, and therefore we may determine λ by writing $\mu V + V_1 = \nu U$; and, equating coefficients, this gives

$$\nu = 1, \quad \frac{y_2}{x_0} + \frac{\mu x_1}{x_0-x_1} = 0, \quad \mu - \lambda - \frac{y_0}{x_0} = 2\cos\omega;$$

whence

$$\lambda = \frac{x_1-x_0}{x_0x_1}y_2 - \frac{y_0}{x_0} = 2\cos\omega.$$

There is a third indeterminate equation $\dfrac{\lambda y_0}{x_0} = \dfrac{\mu y_0}{x_0-x_1} + 1$, which, combined with the others, leads back to the equation of the circle. The y coordinate of p is therefore

$$y_2 + \lambda x_0 = 2y_2 - \frac{x_0}{x_1}y_2 - y_0 - 2x_0\cos\omega,$$

and the y coordinate of q is $y_2 - \dfrac{x_0}{x_1}y_2$, while their x coordinates are the same. The distance between them is therefore $pq = y_2 - y_0 - 2x_0\cos\omega$, and it is thus independent of the direction of AQ. The theorem is therefore proved.

2. The theorem is important as well as interesting, and it therefore seemed desirable to give an elementary proof of it. But the following trilinear proof by Mr. H. M. TAYLOR is both shorter and more general.

The point C being found as in Mr. MERRIFIELD's Solution, let ABC be the triangle of reference; then the equation to the circumscribed conic is

$$\frac{l}{\alpha} + \frac{m}{\beta} + \frac{n}{\gamma} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The equation to the tangent BX is $\dfrac{\alpha}{l} + \dfrac{\gamma}{n} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2).$

The equations to AR, AQ may be written $a\alpha = 2\Delta$, $\beta = k\gamma$ ...... (3, 4), where k is an arbitrary constant.

From (1) and (4) we obtain as the equation to CP

$$l\beta + (m+nk)\alpha = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Similarly, from (2) and (4), the equation to CQ is found to be

$$\frac{\alpha}{l} + \frac{\beta}{kn} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Let β_1, β_2 be the coordinates of p and q respectively; then

from (3) and (5), $\beta_1 = -\dfrac{m+nk}{l}\cdot\dfrac{2\Delta}{a}$,

and from (3) and (6), $\beta_2 = -\dfrac{nk}{l}\cdot\dfrac{2\Delta}{a}$;

therefore $\beta_2 - \beta_1 = \dfrac{2m\Delta}{al}$ = a constant;

therefore pq is also constant.

2464. (Proposed by J. J. WALKER, M.A.)—From a point in the plane of a conic circumscribing a triangle let lines be drawn to meet its three sides parallel respectively to the diameters bisecting those sides. If the area of the triangle formed by lines joining the feet of these parallels be constant, the locus of the point from which they are drawn will be a similar and coaxial conic.

Solution by MORGAN JENKINS, B.A.

Let T be any point in the locus; and let $T\alpha$, $T\beta$, $T\gamma$, drawn in the given directions, meet the sides BC, CA, AB respectively in α, β, γ.

Then $\Delta\alpha\beta\gamma = \Delta T\beta\gamma + \Delta T\gamma\alpha + \Delta T\alpha\beta$ = a constant;

therefore $lyz + mzx + nxy$ = a constant is the trilinear equation to the locus, l, m, n being constants depending only on the given directions in which $T\alpha$, $T\beta$, $T\gamma$ are drawn.

But by Quest. 2395 (the generalization of a well-known theorem) straight lines drawn through any point on the given conic in the same directions meet the sides of the triangle ABC in collinear points; therefore $lyz + mzx + nxy = 0$ is the equation of the given conic, and therefore the required locus is a conic meeting the given conic only at infinity; whence follows the theorem enunciated above.

[Orthogonal projection from the circle proves the theorems in Questions 2395 and 2464 for the ellipse, after which there cannot be much doubt of their general truth.]

2493. (Proposed by Professor CAYLEY.)—1. Given the conic $U = 0$ (but observe that the *function* U contains implicitly an arbitrary constant factor which is *not* given) and also the conic $U + 1 = 0$, to construct the conic $U + l = 0$, where l is a given constant.

2. Given the conics $U = 0$, $U + 1 = 0$, $V = 0$, $V + 1 = 0$, and the constants θ, k, to construct the conic $\theta U + \theta^{-1} V + 2k = 0$.

Solution by the PROPOSER.

1. The conics $U = 0$, $U + 1 = 0$, $U + l = 0$ are obviously concentric similar and similarly situated conics, and if drawing a line in any direction from the

centre, the radius-vectors for the three conics respectively are r, r', R, then it is easy to see that we have

$$R^2 = lr'^2 + (1-l)r^2.$$

There is no difficulty in constructing geometrically the radius R, and then the conic $U + l = 0$ is given as the concentric similar and similarly situated conic passing through the extremity of this radius.

2. To construct the conic $\theta U + \theta^{-1} V + 2k = 0$. By what precedes, we may construct the two conics $\theta U + k = 0$, $\theta^{-1} V + k = 0$; the four points of intersection of these lie on the required conic $\theta U + \theta^{-1} V + 2k = 0$, and also on the conic $\theta U - \theta^{-1} V = 0$; which last conic is consequently given as a conic passing through the four points in question, and also through the four points of intersection of the given conics $U=0$, $V=0$. But the conic $\theta U - \theta^{-1} V = 0$ being constructed, the conic $\theta U + \theta^{-1} V = 0$ can also be constructed; viz, the tangents of these two conics and of the conics $U = 0$, $V = 0$, at each of the four intersections $U = 0$, $V = 0$, form a harmonic pencil; and we have thus the conic $\theta U + \theta^{-1} V = 0$ a conic passing through four given points, and having at each of these a given tangent. And then finally the required conic $\theta U + \theta^{-1} V + 2k = 0$ is given as a conic concentric similar and similarly situated with the conic $\theta U + \theta^{-1} V = 0$, and passing through the four given points

$$\theta U + k = 0, \quad \theta^{-1} V + k = 0.$$

3. Treating k as an absolute constant but θ as a variable parameter, the envelope of the conic $\theta U + \theta^{-1} V + 2k = 0$, is the quartic curve $UV - k^2 = 0$. This is a curve used by Plücker (in the *Theorie der algebraischen Curven*) for the purpose of showing that the 28 double tangents of a quartic curve may be all of them real. In fact, if $U = 0$, $V = 0$, be ellipses intersecting in four real points; and if, moreover, the implicit constants be such that U is positive for points *without* the first ellipse, V positive for points *within* the second ellipse, then since $UV, = k^2$, is positive for all points of the curve in question, the curve must be wholly situate in the four closed spaces which lie outside the one and inside the other of the two ellipses; consisting therefore of four detached portions. And when k is sufficiently small, then the figure of each portion is that of a concavo-convex lens with its angles rounded off: viz., each such portion has a real double tangent of its own. Any two portions have obviously four real double tangents, and hence the total number of real double tangents is $4 + 6 \times 4, = 28$.

4. A construction has been given by Aronhold (*Berl. Monatsber.*, July, 1864) by which, taking any 7 given lines as double tangents of a quartic curve, the remaining 21 double tangents can be constructed, and which, when the seven given lines are real, leads to a system of 28 real double tangents; but wishing to construct the figure of the 28 real double tangents, it occurred to me that the easier manner might be to construct Plücker's curve $UV - k^2 = 0$, as the envelope of the conic $\theta U + \theta^{-1} V + 2k = 0$, and then to draw the tangents of this curve: the construction is, however, practically one of considerable difficulty, and I have not yet accomplished it.

2593. (Proposed by R. Moon, M.A.)—A cylindrical tube, closed at its lower end, and having an air-tight piston capable of moving freely in the upper part of it, rests vertically upon a solid pier. Above the piston there is a vacuum: below it there is air which at the time t is destitute of velocity, but of which the density varies as ϵ^{x}; where x is the distance of a stratum

from the base of the cylinder. A constant pressure equal to the pressure on the piston of the air below it at the time t is exerted upon the piston downwards. Explain the subsequent motion, the air and piston being supposed without weight.

Solution by the Proposer.

If the received law of pressure, viz., $p = a^2\rho$, holds in the above case, since the pressure of the base of the tube on the air above it must be exactly the same as that which the air above it exerts upon it, each particle of the air at the time t will have a uniform effective force f acting upon it; so that at $t+dt$ each particle will have acquired a velocity fdt, and will have described a space $\frac{1}{2}fdt^2$, at the same time that the density throughout will be exactly the same as before. Hence, at the time $t+2dt$ the velocity acquired will be $2fdt$, and the space described will be $\frac{1}{2}f(2dt)^2$; and so for each successive interval; so that, at the time $t+t_1$ the air will have acquired a velocity ft_1, and will have penetrated the pier to a depth $= \frac{1}{2}ft_1^2$; which is absurd.

It is clear that in the above case $\dfrac{dp}{dx}$ must vanish at the base of the cylinder, and increase in value according to some law as we recede from the base.

2483. (Proposed by C. M. Ingleby, LL.D.)—What is the condition of the possibility of a square number being separable into two constituent squares, in more ways than one?

Solution by the Proposer.

If a given square number be separable into two constituent squares, it must be of the form $\left(\dfrac{p^2+q^2}{pq}\right)^2 r^2$, and then its constituent squares are

$$\left(\frac{p^2-q^2}{pq}\right)^2 r^2 \text{ and } 4r^2.$$

[This has been (in effect) proved by the Solution of Question 1974, on p. 39 of Vol. VI. of the *Reprint*.]

If the given square number be separable thus in n ways, we must have

$$\left(\frac{p_1^2+q_1^2}{p_1q_1}\right)^2 r_1^2 = \left(\frac{p_2^2+q_2^2}{p_2q_2}\right)^2 r_2^2 = \ldots\ldots = \left(\frac{p_n^2+q_n^2}{p_nq_n}\right)^2 r_n^2,$$

or,
$$\frac{p_1^2+q_1^2}{p_1q_1}r_1 = \frac{p_2^2+q_2^2}{p_2q_2}r_2 = \ldots\ldots = \frac{p_n^2+q_n^2}{p_nq_n}r_n.$$

Thus, for instance, if we desire to divide 65^2 into two square numbers, we have $p_1=7$, $q_1=4$, and $r_1=1$, which gives $65^2 = 33^2+56^2$; and also $p_2=3$, $q_2=2$, and $r_2=5$, which gives $65^2 = 25^2+60^2$. All other cases of the separation of 65^2 are imaginary.

2562. (Proposed by A. W. PANTON, B.A.)—Show that the equation of a Cartesian oval, referred to the triple focus as origin, is

$$\left\{x^2+y^2-(\beta\gamma+\gamma\alpha+\alpha\beta)\right\}^2+4\alpha\beta\gamma\left\{2x-(\alpha+\beta+\gamma)\right\}=0,$$

where α, β, γ are the distances of the three single foci from the triple focus.

Solution by W. S. MCCAY, B.A.

It is shown in Salmon's *Higher Plane Curves*, that the equation of a Cartesian oval may be written $S^2 = b^3L$, where S is a circle and b a line; and by taking the origin at the centre of S (the triple focus) and the axis of y parallel to L, we may write it $(x^2+y^2-r^2)^2 = b^3(x-a)$. Now if the line $x+iy=c$ touch [i being, as usual, put for $\sqrt{(-1)}$], the real point (if any) on it is a focus. Substitute for x, then

$$(c^2-2icy-r^2)^2 = b^3(c-a-iy).$$

The condition for equal roots in y, arranged in powers of c, is

$$c^3-2c^2a+r^2c+\tfrac{1}{8}b^3=0.$$

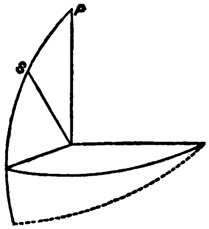
The values of c are the distances from the origin of the foci on the axis of x, the i having disappeared;

therefore $\quad 2c^2=\alpha+\beta+\gamma,\quad r^2=\beta\gamma+\gamma\alpha+\alpha\beta,\quad \tfrac{1}{8}b^3=-\alpha\beta\gamma.$

Putting in these values, we have the equation given in the question.

2421. (Proposed by M. W. CROFTON, B.A.)—Two places are taken at random in the Northern Hemisphere; find the chance of their distance exceeding 90° of a great circle.

Solution by the REV. J. WOLSTENHOLME, M.A.



One point S may be assumed on a given meridian, at a distance θ from the pole. Then, drawing the great circle of which S is pole, we see that the other point will be within 90° of S, unless it lie on a certain line whose area is to that of the hemisphere as $\theta : \pi$; and the chance of S lying between θ and $\theta+d\theta$ is $\sin\theta\, d\theta$. The chance, then, that the two points shall *not* lie within 90° of each other is

$$\frac{1}{\pi}\int_0^{\frac{1}{2}\pi}\theta\sin\theta\,d\theta \equiv \frac{1}{\pi}.$$

2536. (Proposed by C. W. MERRIFIELD, F.R.S.)—When x becomes very large, the number of primes between $x-a$ and $x+a$ approaches the limit $\dfrac{2a}{\gamma+\log x}$, where $\gamma=0{\cdot}57721\ldots$ (Euler's constant.)

Solution by the PROPOSER.

I refer to Euler (*Introductio in Analysin Infinitorum*, Cap. xv.) for the following theorem: If P be the sum of the series of the reciprocals of the prime numbers, and S the harmonic series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\dots$; then $P = \log S + C$, where C is a finite constant. By an unverified calculation, I find

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\dots(\text{primes}) = \log\left\{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\dots(\text{natural nos.})\right\} + 0{\cdot}68428.$$

Moreover the sum of the harmonic series is $= \log x + 0{\cdot}57721\dots + \frac{1}{2x} - \dots$
Now, if x be taken large, the *frequency* of the primes between definite limits x and $x+h$ is measured by the ratio between ΔS_x and ΔP_x, the limiting ratio of which is

$$\frac{d}{dx}S_x : \frac{d}{dx}P_x \text{ or } \frac{d}{dx}(\gamma+\log x) : \frac{d}{dx}\left\{\log(\gamma+\log x)+c\right\} \text{ or } \frac{1}{x} : \frac{1}{x(\gamma+\log x)},$$

whence the theorem.

Ex.: To find the number of primes between 3,034,000 and 3,036,000.

$$\text{Log } 3035000 = 14{\cdot}92572,$$

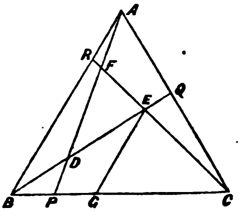
$$\gamma + \log x = 15{\cdot}503 \text{ and } \frac{2000}{15{\cdot}503} = 129.$$

The number counted is 128.

It will be observed that this method does not determine the constant B in the well-known form, when x is large, no. of primes under $x = \frac{x}{A \log x + B}$. It suggests that A should be found = unity, but not why $B = -1{\cdot}08366$.

2601. (Proposed by Capt. A. R. CLARKE, R.E., F.R.S.)—Points P, Q, R, are taken at random on the sides of a triangle ABC; show that the average area of the triangle formed by the intersections of AP, BQ, CR, is $10-\pi^2$.

Solution by S. WATSON; *the* PROPOSER; *and others.*



Let DEF be the triangle, and draw EG parallel to AB. Put $BR = cx$, $CQ = by$; then we easily find

$$EG = \frac{cxy}{x+y(1-x)},$$

$$\therefore \frac{\Delta BEC}{\Delta BAC} = \frac{EG}{AB} = \frac{xy}{x+y(1-x)} \dots\dots (1).$$

Hence the average area of the triangle BEC, in parts of the area of the triangle BAC, is

$$\int_0^1\int_0^1 \frac{xy\,dx\,dy}{x+y(1-x)} = \int_0^1 \left\{\frac{x}{1-x} + \frac{x^2}{(1-x)^2}\log x\right\} dx$$

$$= \int_0^1 \left\{\frac{1-x}{x} + \frac{(1-x)^2}{x^2}\log(1-x)\right\} dx$$

$$= \left\{\frac{1-x^2}{x}\log(1-x) + 2x\right\}_1^0 + \int_1^0 \frac{2}{x}\log(1-x)\,dx$$

$$= -3 + \tfrac{1}{3}\pi^2.$$

By symmetry the average area of each of the triangles CFA, ADB, is the same; hence the average area of DEF is $1-3\left(-3+\frac{1}{3}\pi^2\right) = 10-\pi^2$, [the same result as in Question 2537 and the third part of Question 2557].

2451. (Proposed by Professor CAYLEY.)—If A, B, C, D are the intersections of a conic by a circle, then the antipoints of A, B, and the antipoints of C, D, lie on a confocal conic.

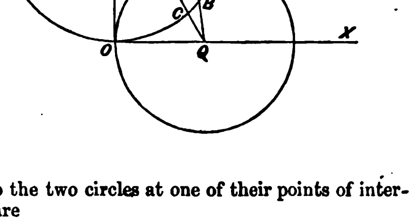
N.B.—If AB, A′B′ intersect at right angles in a point O in such wise that $OA' = OB' = i.OA = i.OB$ [where $i = \sqrt{(-1)}$ as usual], then A′, B′ are the anti-points of A, B, and conversely.

Solution by F. D. THOMSON, M.A.

1. The following construction will be found to give the antifocal points to the four points A, B, C, D on a circle (P):—

"Produce AB, CD to meet in Q, and with Q as centre describe the circle (Q), to cut the circle (P) orthogonally. The points where the perpendiculars from P on AB, CD meet the circle (Q) are the antifocal points required."

2. To show that any conic S round ABCD meets any conic S′ round A′B′C′D′ in four points which lie on a circle.



Take as axes the tangents to the two circles at one of their points of intersection. Then the equations are

$$(P) \equiv x^2+y^2-2by = 0, \qquad (Q) \equiv x^2+y^2-2ax = 0,$$

$$[AB] \equiv y-m(x-a) = 0, \qquad [A'B'] \equiv m(y-b)+x = 0,$$

$$[CD] \equiv y-m'(x-a) = 0, \qquad [C'D'] \equiv m'(y-b)+x = 0;$$

$$\therefore\ S \equiv [AB]\ [CD] + \kappa(P) \equiv \{y - m(x-a)\}\ \{y - m'(x-a)\}$$
$$+\ \kappa\{x^2+y^2-2by\} = 0,$$

and $$S' \equiv [A'B'][C'D'] + \lambda(Q) \equiv \{m(y-b)+x\}\ \{m'(y-b)+x\}$$
$$+\ \lambda\{x^2+y^2-2ax\} = 0;$$

or $$S \equiv y^2-(m+m')xy+mm'x^2+a(m+m')y-2mm\,ax+mm'a^2$$
$$+\kappa\{x^2+y^2-2by\} = 0,$$

and $$S' \equiv mm'y^2+(m+m')xy+x^2-2bmm'y-b(m+m')x+mm'b^2$$
$$+\ \lambda\{x^2+y^2-2ax\} = 0;$$

$$\therefore\ S+S' \equiv (1+mm')(x^2+y^2)+\ \{a(m+m')-2bmm'\}y$$
$$-\{b(m+m')+2amm'\}x+mm'(a^2+b^2)+(\kappa+\lambda)(x^2+y^2)-2b\kappa y-2a\lambda x = 0$$
$$\equiv (1+mm'+\kappa+\lambda)(x^2+y^2)+\{a(m+m')-2bmm'-2b\kappa\}y$$
$$-\{b(m+m')+2amm'+2a\lambda\}x+mm'(a^2+b^2) = 0,$$

which is the equation to a circle passing through the four points of intersection of S and S′.

8. To find the conditions that S and S′ may be confocal.

If $S \equiv (A, B, C, F, G, H)(x, y, 1)^2 = 0$, so that $A = mm'+\kappa$, $B = 1+\kappa$, $C = mm'a^2$, $F = \frac{1}{2}a(m+m')-b\kappa$, $G = -mm\,a$, $H = \frac{1}{2}(m+m')$; the general equation to a conic confocal with $S=0$ is (Salmon's *Conics*, Art. 384)

$$S + \frac{\mu}{\Delta}\Big[(AB-H^2)(x^2+y^2)-2(HF-BG)x-2(GH-AF)y$$
$$+BC-F^2+CA-G^2\Big] + \frac{\mu^2}{\Delta} = 0.$$

Hence, if this coincide with $S' = 0$, that is, with

$$S-\Big[(1+mm'+\kappa+\lambda)(x^2+y^2)+\{a(m+m')-2bmm'-2b\kappa\}y-L\Big] = 0,$$

we must have

$$-\frac{\Delta}{\mu} = \frac{AB-H^2}{1+mm'+\kappa+\lambda} = \frac{-2(GH-AF)}{a(m+m')-2bmm'-2bx} = \frac{2(HF-BG)}{b(m+m')+2amm'+2a\lambda}$$
$$= \frac{BC-F^2+CA-G^2+\mu}{mm'(a^2+b^2)}.$$

Now, substituting the values of A, B, C, &c., and reducing, it will be found that these equations are *consistent*, and that they reduce to

$$-\frac{\Delta}{\mu} = \frac{(1+\kappa)(mm'+\kappa)-\frac{1}{4}(m+m')^2}{1+mm'+\kappa+\lambda} = k; \quad \text{or that} \quad \lambda = -\frac{(m-m')^2}{4\kappa},$$

$$\mu = -\frac{\Delta}{k} = \frac{a^2}{4}(m-m')^2-\kappa\,[mm'(a^2-b^2)+ab(m+m')]+\kappa^2b^2.$$

Substituting these values, we get a conic round A′B′C′D′ confocal with that round ABCD.

2594. (Proposed by C. T. HUDSON, LL.D.)—Let it be required to deal out ab cards n times into a sub-packs of b cards each, and so to arrange the sub-packs after each deal, that after the nth deal and arrangement of the sub-packs any selected card may stand in the rth place from the top of the whole pack.

Solution by the PROPOSER.

1. Suppose the selected card to stand after the first deal sth in its own pack, always counting as first the card which is undermost in any sub-pack, all the cards being dealt face upwards, and the sub-packs when re-arranged into one pack being held by the dealer so that the backs of the cards are towards him.

Let also $p_1, p_2, p_3 \ldots\ldots p_n$ be the places that the sub-pack of the selected card is to hold after the 1st, 2nd, 3rd .... nth regatherings of the sub-packs into one pack. Then after the 1st regathering the selected card will stand in the $b(p_1-1)+s$'th place from the top of the cards, and after the 2nd in the $b(p_2-1)+\dfrac{b(p_1-1)+s+m_1}{a}$'th place from the top of the cards, where m_1 lies between 0 and $a-1$, and must be so taken as to make $b(p_1-1)+s+m$ the least possible multiple of a. After the 3rd regathering the selected card will stand in the

$$b(p_3-1)+\frac{1}{a^2}\Big[b\big\{a(p_2-1)+(p_1-1)\big\}+s+m_1\Big]+\frac{m_2}{a}\text{'th place,}$$

or in the $\dfrac{1}{a^2}\Big[b\big\{a^2(p_3-1)+a(p_2-1)+(p_1-1)\big\}+s+m_1+am_2\Big]$'th place,

where m_1 and m_2 lie between 0 and $a-1$; and after the nth regathering in the

$$\frac{1}{a^{n-1}}\Big[b\big\{a^{n-1}(p_n-1)+a^{n-2}(p_{n-1}-1)+\ldots+a^2(p_3-1)$$
$$+a(p_2-1)+(p_1-1)\big\}+s+m_1+am_2+a^2m_3+\ldots+a^{n-2}m_{n-1}\Big]\text{'th.}$$

Now $\dfrac{1}{a^{n-1}}\big\{m_1+am_2+a^2m_3+\ldots+a^{n-2}m_{n-1}\big\}$ lies between

0 and $\dfrac{1}{a^{n-1}}(a-1)(1+a+a^2+\ldots+a^{n-2})$, or between 0 and $\dfrac{a^{n-1}-1}{a^{n-1}}$,

and since a is a whole number and positive, this latter limit is always <1, and therefore the card's place, or r, lies between

$$\frac{1}{a^{n-1}}\Big[b\big\{(p_1-1)+a(p_2-1)+a^2(p_3-1)+\ldots+a^{n-1}(p_n-1)\big\}+s\Big]\text{ and}$$
$$\frac{1}{a^{n-1}}\Big[b\big\{(p_1-1)+a(p_2-1)+a^2(p_3-1)+\ldots+a^{n-1}(p_n-1)\big\}+s\Big]+1;$$

and therefore $$p_1+ap_2+a^2p_3+\ldots+a^{n-1}p_n \;\ldots\ldots\ldots\ldots\ldots\ldots (\alpha)$$

lies between $\dfrac{1}{b}\Big\{a^{n-1}r+b\,\dfrac{a^n-1}{a-1}-s\Big\}$ and $\dfrac{1}{b}\Big\{a^{n-1}(r-1)+b\,\dfrac{a^n-1}{a-1}-s\Big\}$;

and since s may be as large as $b-1$ and as small as 1, therefore the expression (α) lies between the nearest integer not greater than $\frac{1}{b}\left\{a^{n-1}r + b\frac{a^n-1}{a-1} - (b-1)\right\}$ ... (β),

and the nearest integer not less than $\frac{1}{b}\left\{a^{n-1}(r-1) + b\frac{a^n-1}{a-1} - 1\right\}$ ... (γ);

therefore, to determine $p_1, p_2, p_3 \ldots p_n$, any integer lying between (β) and (γ) is to be divided n times by a, and the successive remainders will give, as is shown by (α), the quantities required. Should any quotient, or the original integer, be a multiple of a, then a itself must be taken as a remainder.

Ex. 1. Let there be 35 cards to be dealt 4 times into 5 sub-packs, and let it be required that the selected card should be in the 23rd place after the 4th deal. Here $a=5$, $b=7$, $n=4$, $r=33$; and substituting in (β) we find the upper limit to be 744; hence from

$$\left.\begin{array}{r|l} 5 & 744 \\ \hline 5 & 148-4 \\ \hline 5 & 29-3 \\ \hline & 5-4 \end{array}\right\} \text{ we have } \left\{\begin{array}{l} p_1 = 4 \\ p_2 = 3 \\ p_3 = 4 \\ p_4 = 5 \end{array}\right.$$

2. The number of solutions is $\left\{(\beta)-p\right\} - \left\{(\gamma)+q\right\} + 1$, where p and q lie between 0 and $b-1$; and therefore when (β) and (γ) are both whole numbers, and consequently p and q are both zero, the number of solutions is $\frac{a^{n-1}+2}{b}$; but when (β) and (γ) are not whole numbers, the number of solutions is the nearest integer not less than $\frac{a^{n-1}+2}{b}-1$ or than $\frac{a^{n-1}+2}{b}-2$, according as $(p+q)$ lies between 0 and b, or b and $2b$. Of course, in any given case, the number of solutions would be obtained directly from (β) and (γ); but the above show that in no case is a solution possible if $a^{n-1}+2<b$.

3. By substituting ab, 1, $\frac{1}{2}(ab+1)$ in succession for r in (β) or (γ), it can be easily shown that, to bring a selected card to the bottom, top, or middle of the pack, after n deals, its sub-pack must always be placed at the bottom, top, or middle of the whole pack; and the smallest possible value of n is determined by the condition $a^{n-1}+2 =$ or $>b$.

The following is a simple case which leads to a neat and easily remembered formula by means of which many tricks may be performed with an ordinary pack of cards.

Let the number of cards be $a_1a_2a_3 \ldots a_n$; and let the cards be dealt out into $a_1, a_2, a_3 \ldots a_n$ sub-packs in succession. Then, with the same notation as before, the place of the selected card after the first deal will be the $a_2a_3a_4 \ldots a_n(p_1-1)+s$ 'th, and after the second the

$$a_1a_3a_4 \ldots a_n(p_2-1) + a_3a_4 \ldots a_n(p_1-1) + \frac{s}{a_2} \text{ 'th;}$$

or, since s lies between 1 and $a_2a_3a_4 \ldots a_n$, the card's place lies between

$$\text{the } a_3a_4a_5 \ldots a_n\left\{a_1(p_2-1)+(p_1-1)\right\}+1 \text{ 'th}$$

$$\text{and the } a_3a_4a_5 \ldots a_n\left\{a_1(p_2-1)+(p_1-1)+1\right\} \text{ 'th.}$$

After the third deal it lies between

$$\text{the}\ a_4 a_5 \dots a_n \left\{a_1 a_2 (p_3-1) + a_1 (p_2-1) + (p_1-1)\right\} + 1\ \text{'th}$$

$$\text{and the}\ a_4 a_5 \dots a_n \left\{a_1 a_2 (p_3-1) + a_1 (p_2-1) + (p_1-1) + 1\right\}\ \text{'th};$$

and after the nth deal and regathering the two limits coincide; and then

$$r = 1 + (p_1-1) + a_1 (p_2-1) + a_1 a_2 (p_3-1) + \dots + a_1 a_2 a_3 \dots a_{n-1} (p_n-1),$$

and therefore $p_1 + a_1 p_2 + a_1 a_2 p_3 + \dots + a_1 a_2 a_3 \dots a_{n-1} p_n$

$$= r + a_1 + a_1 a_2 + a_1 a_2 a_3 + \dots + a_1 a_2 a_3 \dots a_{n-1} \dots (\delta);$$

and $p_1, p_2, p_3, \dots p_n$ may be found by dividing the right-hand member successively by $a_1, a_2, a_3 \dots a_n$; the remainders will be the quantities required.

If there are always at least 3 sub-packs and three deals, then the number of cards that may be used, by the above method, out of an ordinary pack of 52 cards, is $3 \times 3 \times 3$, $3 \times 3 \times 4$, $3 \times 3 \times 5$, or $3 \times 4 \times 4$, *i. e.*, 27, 36, 45, or 48.

In the first three cases, if the deals into 3 sub-packs are taken first, and those into 4 or 5 sub-packs last, the number to be added to r in (δ) will in each case be $3 + 3 \times 3$ or 12.

Ex. 2. Let there be 45 cards to be dealt out twice into 3 sub-packs of 15 each, and once into 5 sub-packs of 9 each, and let it be required to bring any selected card into the 41st place from the top after the third regathering of the cards.

Here $41 + 12 = 53$,

$$\text{and from}\qquad \left.\begin{array}{r|l} 3 & 53 \\ \hline 3 & 17-2 \\ \hline & \;\;5-2 \end{array}\right\}\ \text{we have}\ \left\{\begin{array}{l} p_1 = 2 \\ p_2 = 2 \\ p_3 = 5 \end{array}\right.$$

an operation that can be readily performed mentally.

In the usual method given for doing this trick, the number of cards is always 27, and a long table of the proper values of p_1, p_2, p_3, necessary to produce the required result, must be committed to memory.

2434. (Proposed by Professor WHITWORTH.)—1. Three different persons have each to name an integer not greater than n. The chance that the integers named are proportional to the sides of some real triangle is

$$\tfrac{1}{4}\left(1 + \frac{1}{n^2}\right).$$

2. A person names a group of three integers (not necessarily different, but each one not greater than n). The chance that the integers named are proportional to the sides of some real triangle is

$$\tfrac{1}{2}\left\{1 + \frac{3(n+1)}{2n^2 + 4n + 1 + (-1)^n}\right\}.$$

Solution by SAMUEL ROBERTS, M.A.

Suppose that a, b, c are arranged according to magnitude, and all different. If these quantities are proportional to the sides of a real triangle, we have $c > a-b < b$, or $b > \frac{1}{2}a$. In the case of integers, we have the possible cases

$$\begin{array}{c}\text{when}\\ a\\ \text{is even}\end{array}\left\{\begin{array}{rl} a, & \frac{1}{2}a+1,\ \frac{1}{2}a \\ & \frac{1}{2}a+2,\ \frac{1}{2}a+1 \\ & \qquad\qquad\ \ \frac{1}{2}a \\ & \qquad\qquad\ \ \frac{1}{2}a-1 \\ & \frac{1}{2}a+3,\ \frac{1}{2}a+2 \\ & \qquad\qquad\ \ \frac{1}{2}a+1 \\ & \qquad\qquad\ \ \frac{1}{2}a \\ & \qquad\qquad\ \ \frac{1}{2}a-1 \\ & \qquad\qquad\ \ \frac{1}{2}a-2 \end{array}\right| \qquad \begin{array}{c}\text{when}\\ a\\ \text{is odd}\end{array}\left\{\begin{array}{rr} a, & \frac{1}{2}(a+3),\ \frac{1}{2}(a+1) \\ & \frac{1}{2}(a-1) \\ & \frac{1}{2}(a+5),\ \frac{1}{2}(a+3) \\ & \frac{1}{2}(a+1) \\ & \frac{1}{2}(a-1) \\ & \frac{1}{2}(a-3) \\ & \text{\&c., \&c.} \end{array}\right.$$

$$\text{\&c., \&c.}$$

The number of such combinations, when a is even, is $1+3+5+\ldots+a-3$ or $\frac{1}{4}(a-2)^2$; and when a is odd, the number is $2+4+6+\ldots+a-3$, or $\frac{1}{4}(a-1)(a-3)$. In the present question the limit of a is n. Suppose n even, and we have

$$2\left\{1^2+2^2+\ldots+\tfrac{1}{4}(n-2)^2\right\}-\left\{1+2+\ldots+\tfrac{1}{2}(n-2)\right\},$$

or $$\tfrac{1}{24}(2n^3-9n^2+10n)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (\mathrm{A}).$$

In like manner, when n is odd, we have $\frac{1}{24}(2n^3-9n^2+10n-3)\ \ldots\ldots\ (\mathrm{A}')$.

Similarly, for combinations of the form (a, a, b), we get for n even

$$\tfrac{1}{4}(3n^2-4n)\ \ldots\ldots\ (\mathrm{B}),\quad \text{and for } n \text{ odd }\ \tfrac{1}{4}(3n^2-4n+1)\ \ldots\ldots\ (\mathrm{B}');$$

and the number of cases of (a, a, a) is n in both cases.

As to the *first* part of the question, the total number of combinations is n^3; and we have for the required probability

$$\frac{2\,.\,3\,(\mathrm{A})+3\,(\mathrm{B})+n}{n^3}\ (n \text{ even}),\quad \text{and}\quad \frac{2\,.\,3\,(\mathrm{A}')+3\,(\mathrm{B}')+n}{n^5}\ (n \text{ odd}),$$

which are both comprehended under the form given.

As to the *second* part of the question, whether n is even or odd, the total number of cases is $\frac{1}{6}\left\{n(n+1)(n+2)\right\}$, and the required probability is

$$\frac{(\mathrm{A})+(\mathrm{B})+n}{\frac{1}{6}n(n+1)(n+2)}\ (n \text{ even}),\quad \text{and}\quad \frac{(\mathrm{A}')+(\mathrm{B}')+n}{\frac{1}{6}n(n+1)(n+2)}\ (n \text{ odd});$$

which are both comprehended under the form given.

[Prof. WHITWORTH obtains the following corollary, by supposing $n = \infty$. If three numbers be named at random, they are *as likely as not* to be proportional to the sides of some triangle.]

2558. (Proposed by C. W. MERRIFIELD, F.R.S.)—Mr. Henry Goodwin published in 1818 a table, in which all proper fractions (reduced to their lowest terms) in which the denominator did not exceed 100, nor the numerator 50, were arranged in order of magnitude. He observed the following property, a *general* proof of which is required. Let any three consecutive fractions be $\frac{N_1}{D_1}, \frac{N_2}{D_2}, \frac{N_3}{D_3}$, then $\frac{N_1+N_3}{D_1+D_3} = \frac{N_2}{D_2}$.

I. *Solution by the* PROPOSER.

The difference between two successive terms, $\frac{D_1N_2-D_2N_1}{D_1D_2}$, is always equal to $\frac{1}{D_1D_2}$; for if it were not, and if $D_1N_2-D_2N_1 = k$, where k is an integer greater than unity, we might solve $D_1N_2-D_2N_1 = 1$ as an indeterminate equation yielding an intermediate value, by means of which we could interpolate a term, which is contrary to hypothesis. Hence we have

$$\frac{N_2}{D_2}-\frac{N_1}{D_1}=\frac{1}{D_1D_2}, \text{ and } \frac{N_3}{D_3}-\frac{N_2}{D_2}=\frac{1}{D_2D_3}, \quad \text{whence } \frac{N_2}{D_2}=\frac{N_1+N_3}{D_1+D_3}.$$

II. *Solution by* MORGAN JENKINS, B.A.

To make the proof general, let D be the superior limit of the denominators, $\frac{N_m}{D_m}$ any one of the fractions expressed in its lowest terms, so that N_m is prime to D_m which is less than D. Let ν_m, δ_m be the least positive solutions of

$$D_m\nu - N_m\delta = +1 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1);$$

then, if we take $N_{m+1} = \nu_m + tN_m$, $D_{m+1} = \delta_m + tD_m$ ....(2),

where $t = \mathrm{I}\left(\frac{D-\delta_m}{D_m}\right)$, *i. e.* the greatest integer in the fraction, $\frac{N_{m+1}}{D_{m+1}}$ will be the consecutive fraction of the series in ascending order of magnitude. For if $\frac{P}{Q}$ be any other fraction of the series, such that $PD_m - QN_m = r$ (positive), then

$$P = r\nu_m + sN_m, \quad Q = r\delta_m + sD_m.$$

Hence, if $r > 1$, then since Q is $< D$, sD_m is $< D - r\delta_m < D - \delta_m < tD_m$;

and therefore $s < t$ and $r\delta_m + sD < r(\delta_m + tD_m)$.

Also, if $r = 1$, s is $< t$, as before, and $\delta_m + sD_m$ is $< \delta_m + tD_m$;

therefore in either case $\frac{N_{m+1}}{D_{m+1}} - \frac{N_m}{D_m}$ is $< \frac{P}{Q} - \frac{N_m}{D_m}$;

and this proves that $\frac{N_{m+1}}{D_{m+1}}$ is the next fraction in ascending order of magnitude. In like manner we shall have

$$N_{m-1} = \nu'_m + uN_m, \quad D_{m-1} = \delta'_m + uD_m \quad\ldots\ldots\ldots\ldots(3),$$

where $\nu'_m = N - \nu_m$, and $\delta' = D_m - \delta_m$, $u = I\left(\frac{D-\delta'_m}{D_m}\right)$;

therefore $N_{m+1} + N_{m-1} = vN_m$, $D_{m+1} + D_{m-1} = vD_m$ ....(4),

where $v = 1 + t + u = I\left(\frac{D+D_{m-1}}{D_m}\right) = I\left(\frac{D+D_{m+1}}{D_m}\right)$;

and this proves the theorem for a series of fractions between any given limits $\frac{N_1}{D_1}$ and $\frac{N_l}{D_l}$ (D_1 and D_l both less than D).

Equations (2) must be used to obtain the fraction next greater than $\frac{N_1}{D_1}$; or (3) to obtain the fraction next less than $\frac{N}{D_l}$; and then (4) to complete the series from any consecutive pair of fractions.

2323. (Proposed by S. Roberts, M.A.)—Determine the order, in the coefficients, of the conditions that three cubic equations may have a common root, the same being one of a pair of equal roots as to two of the given equations.

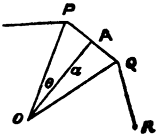
Solution by the Proposer.

There are four conditions represented by five equations of the degrees 2, 2, 2, 2, 3, in x, y, and of the same first order as to the coefficients. We have therefore to take $lmnpq\,\Sigma\frac{1}{lmnp}$ or Σl, substituting the above numbers for l, m, n, p, q. The result is 11. (Salmon's *Higher Algebra*, p. 281.)

2433. (Proposed by W. S. B. Woolhouse, F.R.A.S.)—Five points being taken at random on the surface of a regular polygon of n sides as the apices of a pentagon, show that the probability of two of them being re-entrant is

$$\frac{5}{12}\left\{\frac{\cos\frac{2\pi}{n}+2}{n\sin\frac{2\pi}{n}}\right\}^2.$$

Solution by the Proposer.



Let PQR &c. be a portion of the boundary of the given polygon. By one of the general theorems stated in Question 2471,* if the axes of coordinates be principal axes drawn through the centre O, and h, k denote the radii of gyration of the polygon round those axes, then

$$(\Delta^2) = \frac{3}{2}\frac{h^2k^2}{M^2}, \text{ and } p = 10(\Delta^2) = \frac{15h^2k^2}{M^2},$$

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* *Reprint*, Vol. VIII., p. 100.

where (Δ^2) is the average square of area of an arbitrary triangle drawn on the surface, expressed in parts of M^2, the square of the total area, and p the sought probability.

Now, as the figure is symmetrical, all rectangular axes drawn through the centre are principal axes, and $h=k$. Also, if l denote the radius of gyration round an axis through O perpendicular to the plane of the figure, then $l^2=h^2+k^2$. But if the radii OP, OQ, &c. be drawn, the polygon is composed of n equal triangles symmetrically placed round the centre, and therefore l is the same as the radius of gyration of any one of the triangles (POQ) about the same axis. Take OA bisecting PQ as the axis of x, and let $OA=a$, the angle $POA=\theta$, and the area of the triangle $POQ=m$; then $m=a^2\tan\theta$, and

$$m\,.\,l^2=\Sigma\rho^2 dm=\iint dx\,dy\,(x^2+y^2)=\int dx\,(x^2y+\tfrac{1}{3}y^3)$$

$$\text{(limits } y=-x\tan\theta\ldots+x\tan\theta)$$

$$=2\tan\theta\int x^3dx\,(1+\tfrac{1}{3}\tan^2\theta)=\tfrac{1}{2}a^4\tan\theta\,(1+\tfrac{1}{3}\tan^2\theta)\,;$$

therefore $l^2=\frac{1}{2}a^2(1+\frac{1}{3}\tan^2\theta)$, and $h^2=k^2=\frac{1}{4}a^2(1+\frac{1}{3}\tan^2\theta)$.

Hence, since $M=n\,.\,m=na^2\tan\theta$, we obtain, by the foregoing formula,

$$(\Delta^2)=\frac{3}{32}\left(\frac{1+\frac{1}{3}\tan^2\theta}{n\tan\theta}\right)^2=\frac{1}{24}\left(\frac{3+\tan^2\theta}{2n\tan\theta}\right)^2=\frac{1}{24}\left(\frac{\cos 2\theta+2}{n\sin 2\theta}\right)^2,$$

$$p=10(\Delta^2)=\frac{5}{12}\left(\frac{\cos 2\theta+2}{n\sin 2\theta}\right)^2=\frac{5}{12}\left\{\frac{\cos\frac{2\pi}{n}+2}{n\sin\frac{2\pi}{n}}\right\}^2.$$

As examples, take $n=3, 4, 6, \infty$, and we obtain

for the triangle, $p=\frac{5}{12}\left(\frac{1\frac{1}{2}}{\frac{3}{2}\sqrt{3}}\right)^2=\frac{5}{36}$;

for the square, $p=\frac{5}{12}\left(\frac{2}{4}\right)^2=\frac{5}{48}$;

for the hexagon, $p=\frac{5}{12}\left(\frac{2\frac{1}{2}}{3\sqrt{3}}\right)^2=\frac{125}{1296}$;

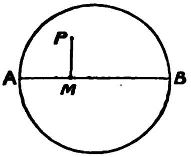
for the circle, $p=\frac{5}{12}\left(\frac{3}{2\pi}\right)^2=\frac{45}{48\pi^2}$.

2500. (Proposed by T. SAVAGE, M.A.)—1. If two points be taken at random within a circle, the chance that the line joining them will meet a given diameter at some point within the circle is $\frac{3\pi^2+16}{6\pi^2}$.

2. If three points be taken at random within a circle, the chance that these three points and the centre of the circle will form the angles of a re-entrant quadrilateral is $\frac{3\pi^2+16}{12\pi^2}$.

Solution by the **Rev. J. Wolstenholme, M.A.**; **S. Watson**; *and others.*

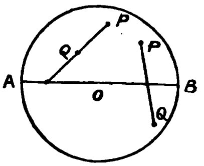
1. Let AB $(=2a)$ be the given diameter, P and Q the two points; then PQ will intersect AB in all cases when P and Q lie on opposite sides of AB. The chance of this is $\frac{1}{2}$. Now let x be the chance that PQ intersects AB when P and Q are on the same side; the whole chance will then be $\frac{1}{2}(1+x)$. But PQ will intersect AB if P lie within the triangle AQB, or Q within the triangle APB;



therefore
$$x = \frac{2 \times \text{average value of the } \triangle \text{APB}}{\text{semicircle}}$$
$$= \frac{\text{AB} \times \text{average value of the ordinate PM}}{\text{semicircle}}$$
$$= \frac{2}{\pi a^2} \cdot 2a \cdot \frac{4a}{3\pi} = \frac{16}{3\pi^2};$$

therefore the whole chance required is $\dfrac{3\pi^2+16}{6\pi^2}$.

2. One of the three points may be considered fixed on the circumference without loss of generality; if we take this point at A, O the centre, P and Q the other two points; then, for a re-entrant quadrilateral, we must have either (α) P and Q on the same side of AB and PQ produced intersecting AO, or (β) P and Q on opposite sides of AB and PQ intersecting BO.

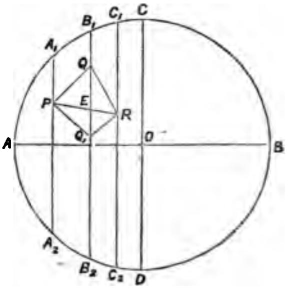


The chance of (α) is $\frac{1}{2} \cdot \frac{1}{2} \cdot x$, and of (β) is $\frac{1}{4}$; therefore the whole chance is $\frac{1}{4}(1+x)$ or $\dfrac{3\pi^2+16}{12\pi^2}$.

2422. (Proposed by the Editor.)—Find the average area of all the triangles that can be drawn within the circumference of a given circle, and deduce therefrom the probability that four points taken at random within the circle will form the apices of a convex quadrangle.

Solution by S. Bills; T. Savage, M.A.; *and others.*

In the accompanying diagram, let ACBD be a circle whose radius is unity, AB and CD being two diameters perpendicular to each other, intersecting in the centre O. Let PQR be one of the inscribed triangles, and through its vertices draw the lines A_1A_2, B_1B_2, C_1C_2, parallel to CD. It is obvious that we shall not destroy the generality of the question by taking the radius equal to unity, and it will greatly simplify the operations. Let Q_1 be the position of Q when below PR; and suppose B_1B_2 to intersect PR in E. Put $A_1P=x'$, $B_1Q=y'$, $C_1R=z'$; angle



$AQA_1 = \theta_1$, $AOB_1 = \theta_2$, $AOC_1 = \theta_3$; then $A_1A_2 = 2\sin\theta_1 = 2s_1$, $B_1B_2 = 2\sin\theta_2 = 2s_2$, $C_1C_2 = 2\sin\theta_3 = 2s_3$; also the perpendiculars from P, Q, R on CD will be, respectively, $\cos\theta_1 = c_1$, $\cos\theta_2 = c_2$, $\cos\theta_3 = c_3$. Now putting $B_1E = v$, we shall have

$$\frac{(s_1-x')-(s_3-z')}{(s_1-x')-(s_2-v')} = \frac{c_1-c_3}{c_1-c_2},$$

therefore $$v = \frac{(c_1-c_2)\,z' + (c_2-c_3)\,x' - (c_1-c_2)\,s_3 - (c_2-c_3)\,s_1}{c_1-c_3} + s_2.$$

Now when Q is *above* PR, we have area $PQR = \frac{1}{2}(c_1-c_3)(v-y')$,
and when Q_1 is *below* PR, we have area $PQ_1R = \frac{1}{2}(c_1-c_3)(y'-v)$.

Hence, $$\frac{1}{2}(c_1-c_3)\left\{\int^{v}(v-y')\,dy' + \int^{2s_2}(y'-v)\,dy'\right\}$$

$$= \frac{1}{2}(c_1-c_3)\left\{(v-s_2)^2+s_2^2\right\} \;\ldots\ldots\ldots\ldots (1)$$

expresses the sum of the areas of PQR while Q ranges from B_1 to B_2.

Restoring the value of v as given above, we have

$$\int_0^{2s_1}\int_0^{2s_3}(1)\,dx'\,dz'$$

$$= \frac{\frac{2}{3}(c_1-c_2)^2 s_1s_3^3 + \frac{2}{3}(c_2-c_3)^2 s_1^3s_3 - 2(c_1-c_3)^2 s_1s_3c_2^2 + 2(c_1-c_3)^2 s_1s_3}{c_1-c_3} \;\ldots\ldots (2),$$

which expresses the sum of the areas of PQR, while P, Q, R range, respectively, over the lines A_1A_2, B_1B_2, C_1C_2. Now put

$$\tfrac{2}{3}s_1s_3^3 + \tfrac{2}{3}s_1^3s_3 - 2s_1s_3(c_1-c_3)^2 = L, \quad \tfrac{4}{3}c_1s_1s_3^3 + \tfrac{4}{3}c_3s_1^3s_3 = M,$$
$$\tfrac{2}{3}c_1^2s_1s_3^3 + \tfrac{2}{3}c_3^2s_1^3s_3 + 2s_1s_3(c_1-c_3)^2 = N;$$

then (2) will become $$\frac{Lc_2^2 - Mc_2 + N}{c_1-c_3} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Also $$\int_{c_3}^{c_1}(3)\,dc_2 = \frac{\frac{1}{3}L(c_1^3-c_3^3) - \frac{1}{2}M(c_1^2-c_3^2) + N(c_1-c_3)}{c_1-c_3}$$

$$= \tfrac{1}{3}L(c_1^2+c_1c_3+c_3^2) - \tfrac{1}{2}M(c_1+c_3) + N \;\ldots\ldots\ldots\ldots (4)$$

will evidently express the sum of the areas of PQR while Q ranges over the entire band $A_1A_2C_2C_1A_1$. Now since the lines A_1A_2, B_1B_2, C_1C_2 may be interchanged in six different ways for the total sum (S suppose) of the areas, when P, Q, R range over the entire surface of the circle, will be

$$S = 6\int_0^{\pi}\int_0^{\theta_1}(4)\,.\,d(1-\cos\theta_3)\,.\,d(1-\cos\theta_1)$$

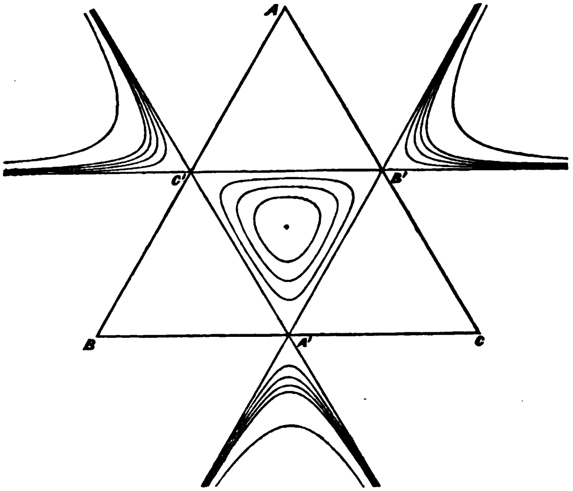
$$= 6\int_0^{\pi}\int_0^{\theta_1}(4)\,s_3s_1\,d\theta_3\,d\theta_1 \;\ldots\ldots\ldots\ldots (5).$$

Restoring the values of L, M, N in (4) and substituting in (5); and then integrating by aid of the usual trigonometrical formula, we find $S = \frac{35}{48}\pi^2$.

Now the number of triangles may evidently be expressed by the cube (π^3) of the area of the circle, since each of the points P, Q, R may range over the entire surface. Hence the required average is $\frac{35\pi^2}{48} \div \pi^3 = \frac{35}{48\pi}$, for a circle of radius unity; and for a radius r it is $\frac{35r^2}{48\pi}$. The probability (p) of the inscribed quadrilateral being re-entrant, is therefore (see *Reprint*, Vol. VI., p. 52) four times this average divided by the area of the circle, or $p = \frac{35}{12\pi^2}$, which agrees with the result otherwise obtained on pp. 88 and 105 of Vol. VIII. of the *Reprint*. It is easy to see that the same result would be obtained for the ellipse.

2557. (Proposed by the EDITOR.)—Show (1) that the locus of the centre of an ellipse of constant area touching the three sides of a triangle is a cubic hyperbola; and, supposing the centres to be equably distributed over every possible position within the triangle, show (2) that the average area of all the inscribed ellipses, in parts of the area of the triangle, is $\frac{4}{105}\pi^2$; moreover, on the same supposition of equable distribution of centres, show (3) that the average area of the triangle formed by joining the points of contact of the inscribed ellipses, in parts of the area of the triangle, is $10-\pi^2$.

I. *Solution by the* REV. J. WOLSTENHOLME, M.A.; *and* CAPT. CLARKE, F.R.S.



1. Taking two sides BC ($=a$) and BA ($=c$) of the triangle for axes of Cartesian coordinates (x and y), the equation of an inscribed conic is

$$\frac{x}{h}+\frac{y}{k}-1=2\lambda\left(\frac{xy}{hk}\right)^{\frac{1}{2}},$$

with the condition [that it should touch AC, viz.],

$$\frac{\lambda^2}{hk}=\left(\frac{1}{h}-\frac{1}{a}\right)\left(\frac{1}{k}-\frac{1}{c}\right)\dots\dots\dots\dots\dots\dots(1).$$

The centre is immediately found to be $\frac{X}{h}=\frac{Y}{k}=\frac{1}{2(1-\lambda^2)}$;

and the equation of the conic referred to its centre is

$$\frac{x^2}{h^2}+\frac{2xy}{hk}(1-2\lambda^2)+\frac{y^2}{k^2}=\frac{\lambda^2}{1-\lambda^2}.$$

If this touch the circle $x^2+2xy\cos B+y^2=r^2$, we have

$$\left(\frac{r^2}{h^2}-\frac{\lambda^2}{1-\lambda^2}\right)\left(\frac{r^2}{k^2}-\frac{\lambda^2}{1-\lambda^2}\right)=\left(\frac{1-2\lambda^2}{hk}r^2-\frac{\lambda^2\cos B}{1-\lambda^2}\right)^2;$$

and the area (πn^2) is $\frac{\pi hk\lambda\sin B}{2(1-\lambda^2)^{\frac{3}{2}}}$;

whence $n^2 = 2XY\lambda(1-\lambda^2)^{\frac{1}{2}} \sin B$ ....(2);

and by (1), $\lambda^2 = \left\{1 - \frac{2X(1-\lambda^2)}{a}\right\}\left\{1 - \frac{2Y(1-\lambda^2)}{c}\right\}$,

or $1 - \frac{2X}{a} - \frac{2Y}{c} + \frac{4XY}{ac}(1-\lambda^2) = 0$ ....(3).

Eliminating λ from (1) and (3), we have

$$\left(1 - \frac{2X}{a} - \frac{2Y}{c} + \frac{2XY}{ac}\right)^2 = \frac{4X^2Y^2}{a^2c^2}\left(1 - \frac{n^4}{X^2Y^2\sin^2 B}\right),$$

or $$\left(1 - \frac{2X}{a} - \frac{2Y}{c}\right)\left(1 - \frac{2X}{a}\right)\left(1 - \frac{2Y}{c}\right) + \frac{4n^4}{a^2c^2\sin^2 B} = 0.$$

Now, if we refer to triangular coordinates (α, β, γ), taking as fundamental triangle A′B′C′, whose vertices are at the middle points of the sides of the original triangle (area = Δ suppose), we have

$$\alpha = \frac{2X}{a} + \frac{2Y}{c} - 1, \quad \beta = 1 - \frac{2X}{a}, \quad \gamma = 1 - \frac{2Y}{c};$$

and the equation of the required locus is $\alpha\beta\gamma = \frac{4n^4}{a^2c^2\sin^2 B} = \frac{n^4}{\Delta^2}$ ......(4).

[The complete locus denoted by (4) is, for various values of n, a series of cubic hyperbolas lying between the asymptotes B′C′, C′A′, A′B′, as represented in the diagram. Of the outer hyperbolic branches, which are the loci of the centres of the escribed ellipses, the four inner individuals correspond respectively to the three closed interior curves and the isolated point the centre, this point being the position of the centre of the greatest inscribed ellipse.]

2. The average area, on the supposition of equable distribution of centres, of all inscribed ellipses lying within the triangle ABC, and whose centres therefore lie within A′B′C′, is, in parts of the area of ABC,

$$\pi\int_0^1\int_0^{1-\alpha}\{\alpha\beta(1-\alpha-\beta)\}^{\frac{1}{2}}\,d\alpha\,d\beta + \int_0^1\int_0^{1-\alpha}d\alpha\,d\beta$$

$$= \frac{\pi^2}{4}\int_0^1 \alpha^{\frac{1}{2}}(1-\alpha)^2\,d\alpha = \frac{4\pi^2}{105}.$$

3. If the equation of the conic (referred to ABC) be

$$(p\alpha')^{\frac{1}{2}} + (q\beta')^{\frac{1}{2}} + (r\gamma')^{\frac{1}{2}} = 0,$$

the area of the triangle formed by joining the points of contact is, in parts of the triangle ABC,

$$1 - \frac{p^2}{(p+q)(p+r)} - \frac{q^2}{(q+r)(q+p)} - \frac{r^2}{(r+p)(r+q)} = \frac{2pqr}{(q+r)(r+p)(p+q)},$$

and the centre of this conic is $\frac{\alpha'}{q+r} = \frac{\beta'}{r+p} = \frac{\gamma'}{p+q}$,

or, referring to triangle A′B′C′,

$$\frac{\beta+\gamma}{q+r} = \frac{\gamma+\alpha}{r+p} = \frac{\alpha+\beta}{p+q}, \text{ or } \frac{\alpha}{p} = \frac{\beta}{q} = \frac{\gamma}{r}.$$

Hence the area of the triangle joining the points of contact is $\frac{2\alpha\beta\nu}{(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)}$; and the average area of this triangle, on the supposition of equable distribution of centres, is

$$\int_0^1\int_0^{1-\alpha}\frac{2\alpha\beta\,(1-\alpha-\beta)\,d\alpha d\beta}{(1-\alpha)\,(1-\beta)\,(\alpha+\beta)}+\int_0^1\int_0^{1-\alpha}d\alpha d\beta$$

$$=4\int_0^1\frac{\alpha d\alpha}{1-\alpha}\int_0^{1-\alpha}\left\{1-\frac{\alpha}{1+\alpha}\left(\frac{1}{\alpha+\beta}+\frac{1}{1-\beta}\right)\right\}d\beta$$

$$=4\int_0^1\frac{\alpha d\alpha}{1-\alpha}\left\{1-\alpha+\frac{2\alpha}{1+\alpha}\log\alpha\right\}$$

$$=2+8\int_0^1(\alpha^2+\alpha^4+\ldots)\log\alpha d\alpha=2-8\left(\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots\right)$$

$$=2-8\left(\frac{\pi^2}{8}-1\right)=10-\pi^2.$$

II. *Solution by* Stephen Watson.

1. Take BC ($=a$) and BA ($=c$) as axes, and let D, E, F be the points of contact of the inscribed ellipse with BC, CA, AB. Put BD $=a\alpha$, BF $=c\beta$; then the equation of the ellipse is

$$\left(\frac{y}{c\beta}-1\right)^2+\left(\frac{x}{a\alpha}-1\right)^2+2kxy-1=0,\text{ where }k=\frac{1-2\,(1-\alpha)(1-\beta)}{ac\alpha\beta}\quad(1)$$

and the coordinates of the centre are

$$l=\frac{a\alpha}{1+kac\alpha\beta}=\frac{a\alpha}{2\,(\alpha+\beta-\alpha\beta)},\quad m=\frac{c\beta}{2\,(\alpha+\beta-\alpha\beta)}\quad\ldots\ldots\ldots(2),$$

therefore $$\alpha=1+\frac{c\,(l-\frac{1}{2}a)}{am}\text{ and }\beta=1+\frac{a\,(m-\frac{1}{2}c)}{cl}\quad\ldots\ldots\ldots\ldots(3).$$

Now the area of the ellipse (1) is easily found to be

$$=\frac{\pi\alpha\beta ac\sin\mathrm{B}}{(1+kuc\alpha\beta)^2}\,(1+kac\alpha\beta)^{\frac{1}{2}}(1-kac\alpha\beta)^{\frac{1}{2}}\ldots\ldots\ldots\ldots\ldots(4),$$

and putting this equal the constant $n\pi\sin\mathrm{B}$, we easily find by the aid of (2) and (3), putting now (x, y) for (l, m), that the equation of the locus of the centre is

$$2\,(ay+cx-\tfrac{1}{2}ac)\,(\tfrac{1}{2}c-y)\,(\tfrac{1}{2}a-x)=n^2\quad\ldots\ldots\ldots\ldots\ldots(5),$$

[which is easily seen to be that of a cubic hyperbola, as represented in the diagram to the foregoing solution.]

2. From the above, the area of the ellipse in parts of the triangle is

$$\frac{2^{\frac{3}{2}}\pi}{uc}(ay+cx-\tfrac{1}{2}ac)^{\frac{1}{2}}(\tfrac{1}{2}c-y)^{\frac{1}{2}}(\tfrac{1}{2}a-x)\quad\ldots\ldots\ldots\ldots\ldots(6);$$

also the limits are as given below, and the number of ellipses is expressed by $\frac{1}{8}ac\sin\mathrm{B}$; hence the average area of the ellipses is

$$\frac{8}{uc\sin\mathrm{B}}\int_0^{\frac{1}{2}c}d\,(y\sin\mathrm{B})\int_{(\frac{1}{2}-\frac{y}{c})a}^{\frac{1}{2}a}(6)\,dx=$$

$$-\frac{16\pi}{a^2c^2}\int_0^{\frac{1}{2}c}(c-2y)^{\frac{1}{2}}\,dy\int_{\left(\frac{ay}{c}\right)^{\frac{1}{2}}}^{)}(ay-cu^2)^{\frac{1}{2}}\,2u^2du, \text{ where } u^2=\tfrac{1}{2}a-x,$$

$$=\frac{2\pi^2}{c^{\frac{7}{2}}}\int_0^{\frac{1}{2}c}(c-2y)^{\frac{1}{2}}\,y^2dy=\frac{4\pi^2}{105} \text{ of the triangle ABC.}$$

3. The area of the triangle BDF in parts of that of the triangle ABC is

$$\frac{\text{BD}\ .\ \text{BF}}{ac}=\alpha\beta=\left\{1+\frac{c\,(l-\frac{1}{2}a)}{am}\right\}\left\{1+\frac{a\,(m-\frac{1}{2}c)}{cl}\right\}\ldots\ldots\ldots\ldots(7);$$

hence the average area of the triangle BDF is

$$\frac{8}{ac\sin\text{B}}\int_0^{\frac{1}{2}c}d\,(m\sin\text{B})\int_{\left(\frac{1}{2}-\frac{m}{c}\right)a}^{\frac{1}{2}a}(7)\,dl$$

$$=-\frac{1}{2}-\frac{8}{c^2}\int_0^{\frac{1}{2}c}\frac{(m-\frac{1}{2}c)^2}{m}\log\left(1-\frac{2m}{c}\right)dm$$

$$=-3-\left\{\frac{4}{c^2}\,(m^2-2cm+\tfrac{3}{4}c^2)\log\left(1-\frac{2m}{c}\right)\right\}_0^{\frac{1}{2}c}+2\int_0^{\frac{1}{2}c}\frac{1}{m}\log\left(1-\frac{2m}{c}\right)dm$$

$$=-3+\frac{\pi^2}{3}.$$

The same holds good of the average area of each of the triangles CED, AFE; therefore the average area of the triangle DEF is

$$1-3\left(-3-\frac{\pi^2}{3}\right)\quad 10-\pi^2.$$

2590. (Proposed by Professor Cayley.)—It is required to verify Professor Kummer's theorem that "if a quartic surface is such that every section by a plane through a certain fixed point is a pair of conics, the surface is a pair of quadric surfaces (except only in the case where it is a quartic cone having its vertex at the fixed point)."

Solution by the PROPOSER.

The theorem may be more generally stated as follows; if a surface is such that every section through a certain fixed point (is or) includes a proper conic, then the surface (is or) includes a proper quadric surface. In order to the demonstration, I premise the following *Lemma:*—If a surface is such that every section through a certain fixed line includes a conic, then the line meets each of these conics in the same two points.

In fact, if the line meet the surface in any n points, then it is clear that each of the conics will meet the line in some two of these n points; and as the plane of the section passes continuously from any one to any other position, the two points of intersection with the conic cannot pass abruptly from being some two to being some other two of the n points, that is, they are always the same two points.

Consider now a surface which is such that every section through a fixed point O includes a conic; and consider three lines *xx′*, *yy′*, *zz′* meeting in the point O. Let the conics in the planes *yz*, *zx*, *xy* be A, B, C respectively; then since the conics through the line *xx′* all pass through the same two points, and since B, C are two of these conics, B and C meet *xx′* in the same two points X, X′; similarly C and A meet *yy′* in the same two points Y, Y′; and A, B meet *zz′* in the same two points Z, Z′; that is, we have the conics A, B, C intersecting

B, C in the two points X, X′,
C, A „ „ Y, Y′,
A, B „ „ Z, Z ;

hence taking on the conics A, B, C the points α, β, γ respectively, and drawing a quadric surface Σ through the nine points X, X , Y, Y,′ Z, Z,′ α, β, γ, this meets the conic A in the five points Y, Y′, Z, Z,′ α; that is, it passes through the conic A, and similarly it passes through the conic B, and through the conic C.

Consider how any plane whatever through O intersecting the conics A, B, C in the points L and L′, M and M′, N and N′ respectively; the section of the quadric surface Σ by the plane in question is a conic through the six points L, L′, M, M′, N, N′. But the section of the surface includes a conic through these same six points, and which is consequently the same conic; in fact, the section of the surface by the plane in question includes a conic, and since every section through the line LL′ includes a conic through the same two points, and one of these conics is the conic A which passes through the points L and L′, the conic in question passes through the points L and L′; and similarly it passes through the points M and M , and through the points N and N′. That is, for any plane whatever through O, the section of the surface includes the conic which is the section of the quadric surface Σ, and the surface thus includes as part of itself the quartic surface Σ.

The foregoing demonstration ceases, however, to be applicable if O is a point on the surface, and the conic included in the section through O is always a conic passing through the point O. In the case where O is a non-singular point of the surface (that is, where there is at O a unique tangent plane) a like demonstration applies. Take through O a section, and let this include the conic A; on A take any point O′ and through OO′ a section including the conic B; we have thus the conics A, B intersecting in the points O, O′. Take through O any plane meeting the conics A, B in the points X, Y respectively—the section by this plane includes a conic C passing through the points O, X, Y; and each of the conics A, B, C touches at O the same plane, viz., the tangent plane of the surface. Hence, taking on the conic A the point α, consecutive to O, and any other point α'; on the conic B the point β, consecutive to O, and any other point β'; and on the conic C a point γ'; we may, through the nine points O, α, β, O′, α', β', X, Y, γ' describe a quadric surface Σ; this will touch at O the tangent plane of the surface, that is, it will touch the conic C, or (what is the same thing) pass through a point γ of this conic consecutive to O. Hence the quadric surface meets the conic A in the five points O, O′, α, α', X, that is, it entirely contains the conic A; similarly it meets the conic B in the five points O, O′, B, B′, Y, that is, it entirely contains the conic B; and it meets the conic C in the five points O, γ, X, Y, γ', that is, it entirely contains this conic. And it may then be shown as before that the surface will include the quadric surface Σ. But there still remains for consideration the case where O is a conical point on the surface, and I do not at present see how this is to be treated.

I remark that, taking three lines xx', yy', zz' which meet in a point O, then if a surface be such that every section through xx' includes a conic, every section through yy' includes a conic, and every section through zz' includes a conic; and if besides the two points, say X, X′, on the conics through the line xx' are ordinary points on the surface, then the surface will include a quadric surface. In fact, if the surface has at each of the points X, X′ an ordinary tangent plane, then the conics through xx', and (as conics of the series) the two conics B, C all of them touch the two tangent planes; hence, constructing as before the quadric surface Σ, this also touches the two tangent planes: and taking through xx' a plane meeting the conic A in the points L, L′, the section of the surface includes a conic which touches the section of the quadric surface Σ at the points X, X′, and besides meets it in the points L, L′; such conic coincides therefore with the section of the quadric surface Σ; that is, every section of the surface through the line xx' includes the conic which is the section of the quadric surface Σ; and the surface thus includes as part of itself the quadric surface Σ.

2592. (Proposed by the Rev. J. WOLSTENHOLME, M.A.)—If a finite straight line be marked at random in $p-1$ points and broken up at those points, (1) the chance that none of the pieces is less than one-mth of the whole $(m>p)$ is $\left(1-\frac{p}{m}\right)^{p-1}$; (2) the chance that none of the pieces is greater than one-nth of the whole $(n<p)$ is

$$\left(\frac{p}{n}-1\right)^{p-1}-p\left(\frac{p-1}{n}-1\right)^{p-1}+\frac{p(p-1)}{2}\left(\frac{p-2}{n}-1\right)^{p-1}-\ldots\ldots$$

to r terms, r being the integer next $>p-n$; (3) the chance that none of the pieces is less than one-mth and none greater than one-nth is

$$\left(1-\frac{p}{m}\right)^{p-1}-p\left(1-\frac{p-1}{m}-\frac{1}{n}\right)^{p-1}+\frac{p(p-1)}{2}1-\frac{p-2}{m}-\frac{2}{n}\Big)^{p-1}-\ldots$$

to r terms, where r is the integer next $>n\frac{m-p}{m-n}$; provided that $\frac{1}{m}+\frac{p-1}{n}>1$ and $\frac{1}{n}+\frac{p-1}{m}<1$, otherwise one of the chances includes the other and has been already determined.

Solution by Professor WHITWORTH.

1. The number of ways in which mk *indifferent* things can be distributed into p *different* parcels, so that no parcel contain less than k, is (*Messenger of Mathematics*, No. XV., p. 155) $\frac{\lfloor\underline{(m-p)k+p-1}}{\lfloor\underline{(m-p)k}\,.\,\lfloor\underline{p-1}}$. But the number of ways in which the distribution can be made without limitation is $\frac{\lfloor\underline{mk-1}}{\lfloor\underline{mk-p}\,\lfloor\underline{p-1}}$. Hence if the distribution be made at random, the chance

that no parcel contain less than k is $\dfrac{\lfloor\!\underline{(m-p)k+p-1}\;\lfloor\!\underline{mk-p}}{\lfloor\!\underline{(m-p)k}\;\lfloor\!\underline{mk-1}}$. Now, suppose k be infinite, then the result becomes $\dfrac{(m-p)^{p-1}}{m^{p-1}}$ or $\left(1-\dfrac{p}{m}\right)^{p-1}$ which is therefore the chance that if an infinite series of elements be divided into p parts in order, no part will be less than one-mth of the whole.

2. The number of ways in which nk *indifferent* things can be distributed into p *different* parcels, no parcel containing more than k, is the coefficient of x^{nk} in the expansion of

$$(x+x^2+x^3+\ldots+x^k)^p = \text{co. of } x^{nk-p} \text{ in } (1-x^k)^p(1-x)^{-p}$$

$$=\frac{\lfloor\!\underline{nk-1}}{\lfloor\!\underline{nk-p}\;\lfloor\!\underline{p-1}}-\frac{p}{1}\cdot\frac{\lfloor\!\underline{(n-1)k-1}}{\lfloor\!\underline{(n-1)k-p}\;\lfloor\!\underline{p-1}}+\frac{p(p-1)}{1.2}\cdot\frac{\lfloor\!\underline{(n-2)k-1}}{\lfloor\!\underline{(n-2)k-p}\;\lfloor\!\underline{p-1}}-\&c.,$$

the number of terms being the integral part of n.

But the number of ways in which the parcels can be formed without limitation is $\dfrac{\lfloor\!\underline{nk-1}}{\lfloor\!\underline{nk-p}\;\lfloor\!\underline{p-1}}$. (*Messenger of Mathematics*, as above)

Hence, if the parcels be formed at random, the chance that none is greater than k is

$$1-\frac{p}{1}\cdot\frac{\lfloor\!\underline{nk-p}\;\lfloor\!\underline{(n-1)k-1}}{\lfloor\!\underline{nk-1}\;\lfloor\!\underline{(n-1)k-p}}+\frac{p(p-1)}{1.2}\cdot\frac{\lfloor\!\underline{nk-p}\;\lfloor\!\underline{(n-2)k-1}}{\lfloor\!\underline{nk-1}\;\lfloor\!\underline{(n-2)k-p}}-\&c.$$

Now suppose k infinite, then the result becomes

$$1-\frac{p}{1}\left(1-\frac{1}{n}\right)^{p-1}+\frac{p(p-1)}{1.2}\left(1-\frac{2}{n}\right)^{p-1}-\frac{p(p-1)(p-2)}{1.2.3}\left(1-\frac{3}{n}\right)^{p-1}+\&c.,$$

(the number of terms being n if integral, or the next greater integer), which is therefore the chance that if an infinite series of elements be divided into p parts in order, no part will be greater than one-nth of the whole.

This is the form in which Mr. WOLSTENHOLME proposed the result in the Mathematical Tripos Examination, Jan. 15, 1868.

3. The number of ways in which mnk *indifferent* things can be distributed into p parcels, so that no parcel contain less than nk nor more than mk, is found by supposing $nk-1$ things placed in each of the p parcels, and then considering how the remaining $(m-p)nk+p$ things can be further distributed into the p parcels without putting more than $(m-n)k+1$ into each parcel.

Therefore by the preceding case the result is the coefficient of $x^{(m-p)nk+p}$ in the expansion of $(x+x^2+x^3+\ldots+x^{(m-n)k+1})^p$

$$=\frac{\lfloor\!\underline{(m-p)nk+p-1}}{\lfloor\!\underline{(m-p)nk}\;\lfloor\!\underline{p-1}}-\frac{p}{1}\cdot\frac{\lfloor\!\underline{(m-p)nk-(m-n)k+p-2}}{\lfloor\!\underline{(m-p)nk-(m-n)k-1}\;\lfloor\!\underline{p-1}}+\&c.,$$

the number of terms being $\dfrac{(m-p)nk+p}{(m-n)k+1}$ or the next greater integer.

But the number of ways in which the parcels can be formed without limitation is

$$\frac{\lfloor\underline{mnk-1}}{\lfloor\underline{mnk-p}\,\lfloor\underline{p-1}}.$$

Hence if the parcels be formed at random, the chance that none is less than *nk* or greater than *mk* is

$$\frac{\lfloor\underline{(m-p)nk+p-1}\,\lfloor\underline{mnk-p}}{\lfloor\underline{(m-p)nk}\,\lfloor\underline{mnk-1}}-\frac{p}{1}\cdot\frac{\lfloor\underline{(m-p)nk-(m-n)k+p-2}\,\lfloor\underline{mnk-p}}{\lfloor\underline{(m-p)nk-(m-n)k-1}\,\lfloor\underline{mnk-1}}$$
$$+\,\&c.$$

Now suppose *k* infinite: then the result becomes

$$\left\{\frac{(m-p)n}{mn}\right\}^{p-1}-\frac{p}{1}\left\{\frac{(m-p)n-(m-n)}{mn}\right\}^{p-1}$$
$$+\frac{p\,.\,(p-1)}{1\,.\,2}\left\{\frac{(m-p)n-2(m-n)}{mn}\right\}^{p-1}-\,\&c.,$$

or

$$\left\{1-\frac{p}{m}\right\}^{p-1}-p\left\{1-\frac{p-1}{m}-\frac{1}{n}\right\}^{p-1}+\,\&c.,$$

(the number of terms being the integral part of $\frac{(m-p)n}{m-n}$,) which is therefore the chance that, if an infinite series of elements be divided into *p* parts in order, no part will be less than one-*m*th or greater than one-*n*th of the whole.

ON THE INTEGRATION OF DIFFERENTIAL EQUATIONS.

By Chief Justice Cockle.

1. **Theorem I.** If the coefficients of the two linear differential equations of the second order

$$\frac{d^2y}{dx^2}+2q\frac{dy}{dx}+ry=0,\qquad\frac{d^2\mathrm{Y}}{dx^2}+2\mathrm{Q}\frac{d\mathrm{Y}}{dx}+\mathrm{RY}=0\;\ldots\ldots\ldots(1,2),$$

are connected by the relation $q^2+\frac{dq}{dx}-r=\mathrm{Q}^2+\frac{d\mathrm{Q}}{dx}-\mathrm{R}\;\ldots\ldots\ldots\ldots(3),$

then also are the dependent variables y and Y connected by the relation

$$\mathrm{C}e^{\int(q-Q)\,dx}\,y=\mathrm{Y}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

For, in (2) substitute uy for Y and divide the result by u. We find

$$\frac{d^2y}{dx^2}+2\left\{\frac{1}{u}\left(\frac{du}{dx}\right)+\mathrm{Q}\right\}\frac{dy}{dx}+\left\{\frac{1}{u}\left(\frac{d^2u}{dx^2}\right)+2\mathrm{Q}\frac{1}{u}\left(\frac{du}{dx}\right)+\mathrm{R}\right\}y=0\;\ldots.(5).$$

Now, in order that (5) may be identical with (1), we have the following conditions:

$$\frac{1}{u}\cdot\frac{du}{dx}+\mathrm{Q}=q,\qquad\frac{1}{u}\cdot\frac{d^2u}{dx^2}+2\mathrm{Q}\frac{1}{u}\cdot\frac{du}{dx}+\mathrm{R}=r\;\ldots\ldots\ldots(6,7).$$

Differentiate (6) with respect to the independent variable x, and subtract the result from (7). We have

$$\frac{1}{u^2}\left(\frac{du}{dx}\right)^2+2\mathrm{Q}\frac{1}{u}\cdot\frac{du}{dx}+\mathrm{R}-\frac{d\mathrm{Q}}{dx}=r-\frac{dq}{dx}\ldots\ldots\ldots\ldots(8).$$

Next, in (8), substitute for $\frac{1}{u} \cdot \frac{du}{dx}$ its value $(q-Q)$ deduced from (6). Then (8) becomes

$$(q-Q)^2+2Q(q-Q)+R-\frac{dQ}{dx}=r-\frac{dq}{dx} \;\ldots\ldots\ldots\ldots (9),$$

or

$$q^2-Q^2+R-r+\frac{dq}{dx}-\frac{dQ}{dx}=0 \;\ldots\ldots\ldots\ldots (10),$$

which last equation is equivalent to (3), and the theorem is demonstrated, for if u be determined from (6) the relation $uy = Y$ conducts us to (4).

2. THEOREM II. The binordinary $\frac{d^2y}{dx^2}-\left\{\frac{1}{2r}\cdot\frac{dr}{dx}+a\sqrt{r}\right\}\frac{dy}{dx}+ry=0 \ldots (11)$ is soluble whatever function of x the quantity r may be, for it is transformable into an equation with constant coefficients.

This will most readily appear by changing the independent variable from x to t, which latter quantity is determined by means of

$$\int\sqrt{r}\,.\,dx = Ct+C_2 \;\ldots\ldots\ldots\ldots\ldots\ldots (12),$$

C and C_2 being constants. If the latter constant be supposed to vanish, the transformed equation will be

$$\frac{d^2y}{dt^2}-aC\frac{dy}{dt}+C^2y=0 \;\ldots\ldots\ldots\ldots\ldots\ldots (13),$$

whereof all the coefficients are constant. The relation

$$\left(\frac{d}{dx}-\frac{1}{2r}\cdot\frac{dr}{dx}+\beta\sqrt{r}\right)\left(\frac{d}{dx}+\gamma\sqrt{r}\right)y=0 \;\ldots\ldots\ldots (14),$$

wherein β and γ are determined from

$$\beta+\gamma=a, \quad \beta\gamma=1 \;\ldots\ldots\ldots\ldots\ldots\ldots (15.\ 16),$$

is a symbolical decomposition, and consequently a means of solution, of (11). This decomposition, and in general all such decompositions, will be followed without much difficulty by operating first on y with the right hand factor and then on the result with the left hand factor, and so on for any number of symbolical factors. If we divide (11) by r, the result, viz.,

$$\frac{1}{r}\cdot\frac{d^2y}{dx^2}-\left\{\frac{1}{2r^2}\cdot\frac{dr}{dx}+a\frac{1}{\sqrt{r}}\right\}\frac{dy}{dx}+y=0 \;\ldots\ldots\ldots\ldots (17),$$

is symbolically decomposable into

$$\left(\frac{1}{\sqrt{r}}\cdot\frac{d}{dx}+\beta\right)\left(\frac{1}{\sqrt{r}}\cdot\frac{d}{dx}+\gamma\right)y=0 \ldots\ldots\ldots\ldots (18),$$

where β and γ are determined from (15, 16) as before. Hence we are readily led to (12). For if

$$\frac{1}{\sqrt{r}}\cdot\frac{dy}{dx}=\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt} \;\ldots\ldots\ldots\ldots (19),$$

we have

$$\frac{1}{\sqrt{r}}=\frac{dx}{dt} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (20),$$

which is easily reducible to a case of (12).

3. There are two other forms, soluble for all values of r, and somewhat resembling (11), viz.,

$$\frac{d^2y}{dx^2}-\left(\frac{1}{r}\cdot\frac{dr}{dx}+ar+\frac{1}{a}\right)\frac{dy}{dx}+ry=0 \;\ldots\ldots\ldots\ldots (21),$$

and

$$\frac{d^2y}{dx^2}-\left(\frac{1}{r}\cdot\frac{dr}{dx}+ar+b\right)\frac{dy}{dx}+\frac{b}{r}\cdot\frac{dr}{dx}\cdot y=0 \;\ldots\ldots (22).$$

The symbolical decomposition of (21) is

$$\left(\frac{d}{dx}-\frac{1}{r}\cdot\frac{dr}{dx}-\frac{1}{a}\right)\left(\frac{d}{dx}-ar\right)y=0\ldots\ldots\ldots\ldots(23),$$

and that of (22) is
$$\left(\frac{d}{dx}-\frac{1}{r}\cdot\frac{dr}{dx}\right)\left(\frac{d}{dx}-ar-b\right)y=0\ldots\ldots\ldots\ldots(24).$$

4. If we develope the symbolical equation

$$\left\{(b+cx)\frac{d}{dx}+\frac{r}{a}\right\}\left\{(b+cx)\frac{d}{dx}+a\right\}y=0\ldots\ldots\ldots\ldots(25),$$

we obtain the result $(b+cx)^2\dfrac{d^2y}{dx^2}+\left(\dfrac{r}{a}+a+c\right)(b+cx)\dfrac{dy}{dx}+ry=0$ ...(26),

and this last equation is therefore soluble for any value whatever of r. An easy illustration of symbolical decomposition is afforded by the equation

$$\left(\frac{d}{dx}+b+ax\right)\left(\frac{d}{dx}+b-ax\right)y=0\ldots\ldots\ldots\ldots(27),$$

which, developed, becomes

$$\frac{d^2y}{dx^2}+2b\frac{dy}{dx}+(b^2-a-a^2x^2)y=0\ldots\ldots\ldots\ldots(28).$$

5. *Scholium.*—If we call the functions on either side of equation (3) criticoids, Theorem I. indicates that if an equation having a certain criticoid be soluble, then all equations having the same criticoid are soluble.

6. Let us apply portions of the foregoing discussion to some of the equations of Riccati. And, first, let

$$\frac{d^2y}{dx^2}+\frac{a}{x^2}.y=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(29),$$

an equation which by exponential substitution (*i.e.* by taking $y=e^{\int zdx}$) may be transformed from its present binomial form to the trinomial form considered by Riccati. Now, comparing (29) with (11) we see that the former equation may be solved by change of the independent variable, and (12) suggests the assumption

$$t=\log x\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(30),$$

which, being made, changes (29) into

$$\frac{d^2y}{dt^2}-\frac{dy}{dt}+ay=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(31),$$

an equation with constant coefficients.

7. Next let
$$\frac{d^2y}{dx^2}+\frac{a}{x^4}.y=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(32).$$

In this case, still conforming with (12), let $t=\dfrac{1}{x}$ ....(33),

then $\dfrac{dx}{dt}=-\dfrac{1}{t^2}$ and $\dfrac{d^2x}{dt^2}=\dfrac{2}{t^3}$. Hence (32) becomes

$$\frac{d^2y}{dt^2}+\frac{2}{t}\cdot\frac{dy}{dt}+ay=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(34),$$

and the criticoid of this equation is the same as that of

$$\frac{d^2y}{dt^2}+ay=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(35),$$

an equation with constant coefficients. Consequently, by Theorem I., equation (34) is soluble, and the solution of (32) is in substance reduced to that of (35).

8. If v be an integral of (34) then tv is an integral of (35), and if w be an integral of (35) then $\frac{w}{t}$ is an integral of (34). And it may be convenient to remark here that the solution of

$$\frac{d^2y}{dx^2} + ry = 0 \quad \dots\dots\dots\dots\dots\dots\dots\dots (36)$$

depends upon that of $\frac{d^2z}{dx^2} - \frac{1}{r} \cdot \frac{dr}{dx} \cdot \frac{dz}{dx} + rz = 0 \quad \dots\dots\dots\dots\dots\dots (37)$

and *vice versâ*. For if we divide (36) by r, differentiate the quotient, multiply the result into r, and then replace $\frac{dy}{dx}$ by z, the final equation thus obtained will be (37).

9. Next consider the equation $\frac{d^2y}{dx^2} + \frac{a}{\sqrt[3]{x^4}} y = 0 \quad \dots\dots\dots\dots\dots (38)$,
from which we derive

$$\frac{d^2z}{dx} + \frac{4}{8} \cdot \frac{1}{x} \cdot \frac{dz}{dx} + \frac{a}{\sqrt[3]{x^4}} \cdot z = 0 \quad \dots\dots\dots\dots\dots (39)$$

in the same way that (37) is derived from (36).

Let $$x = \frac{1}{t^3} \dots\dots\dots\dots\dots\dots\dots\dots\dots (40)$$

and consequently $$\frac{dx}{dt} = -\frac{3}{t^4}, \ \frac{d^2x}{dt^2} = \frac{12}{t^5};$$

then (39) may be transformed into

$$\frac{d^2z}{dt^2} + \left\{\frac{4}{3} t^3 \left(-\frac{8}{t^4}\right) + \frac{4}{t}\right\} \frac{dy}{dt} + at^4 \left(\frac{3^2}{t^8}\right) y = 0 \quad \dots\dots\dots (41),$$

or, reducing, into $$\frac{d^2z}{dt^2} + \frac{9a}{t^4} \cdot y = 0 \dots\dots\dots\dots\dots\dots\dots (42),$$

o that the solution of (38) is thus made to depend upon that of (32).

10. Another transformation may be applied to (39). Let

$$x = \frac{\epsilon^{-3ct}}{-3c} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots (43),$$

and therefore $$\frac{dx}{dt} = \epsilon^{-3ct}, \ \frac{d^2x}{dt^2} = -3c\epsilon^{-3ct}$$

then (39) may be transformed into

$$\frac{d^2z}{dt^2} + \left\{\tfrac{4}{3}(-3c) + 3c\right\} \frac{dz}{dt} + a\,(-3c)^{\frac{4}{3}}\, \epsilon^{-2ct} z = 0 \quad \dots\dots\dots (44),$$

which, if we put $$a \sqrt[3]{\left\{(-3c)^4\right\}} = \mathrm{A},$$

becomes $$\frac{d^2z}{dt^2} + c\frac{dz}{dt} + \mathrm{A}e^{-2ct} \cdot z = 0 \dots\dots\dots\dots\dots\dots (45).$$

11. Since (42) is soluble, (45) is soluble, as also is

$$\frac{d^2z}{dt^2}+\left\{Ae^{-2ct}-\frac{c^2}{4}\right\}z=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(46),$$

which has the same criticoid as (45) and in which A and c are any constants whatever. And it may be convenient to remark here that whenever c is constant and

$$\frac{d^2y}{dx^2}+c\frac{dy}{dx}+ry=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(47)$$

soluble, then also the equation

$$\frac{d^2y}{dx^2}-c\frac{dy}{dx}+ry=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(48)$$

is soluble, for (47) and (48) have the same criticoid.

12. Theorems I. and II. are of considerable generality, and it may be interesting to compare Boole's *Differential Equations*, 2nd ed., p. 411, Ex. 11; p. 459, Ex. 14; and pp. 428–429, Art. 6. With Ex. 11 of p. 411, just cited, may be compared the equation

$$\frac{d^2y}{dx^2}+(b+X)\frac{dy}{dx}+bXy=0\ldots\ldots\ldots\ldots\ldots(9),$$

in which X is any function whatever of x. This equation is deducible from (22) by making a vanish and performing an easy substitution. If in (37) we change the independent variable from x to t, and so make the middle term of (37) a constant, say c, we find

$$\int r dx=-\frac{1}{c}\cdot\epsilon^{-ct}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(50),$$

so that if $r=x^m$ we have $\quad x=\left(-\frac{m+1}{c}\right)^{\frac{1}{m+1}}\cdot\epsilon^{-\frac{ct}{m+1}}\ldots\ldots\ldots(51)$

and $\quad r\left(\frac{dx}{dt}\right)^2=\frac{1}{r}\cdot\left(r\frac{dx}{dt}\right)^2$

$$=\epsilon^{-2ct}\cdot\left(-\frac{m+1}{c}\right)^{-\frac{m}{m+1}}\cdot\epsilon^{\frac{mct}{m+1}}=\left(-\frac{m+1}{c}\right)^{-\frac{m}{m+1}}\cdot\epsilon^{-\left(\frac{m+2}{m+1}\right)ct}\ldots(52).$$

Now, applying theorem II., we know that, A being any constant,

$$\frac{d^2z}{dt^2}+c\frac{dz}{dt}+A\epsilon^{-\left(\frac{m+2}{m+1}\right)ct}z=0\ldots\ldots\ldots\ldots\ldots\ldots(53)$$

will be soluble by a change of the independent variable provided that

$$c=\frac{1}{2}\left(\frac{m+2}{m+1}\right)c,\ \text{or}\ m=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(54).$$

Again, as in Art. 11, we infer that when (53) is soluble, then also

$$\frac{d^2z}{dt^2}-c\frac{dz}{dt}+Ae^{-\left(\frac{m+2}{m+1}\right)ct}z=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(55).$$

But, by Theorem II., (55) is soluble when

$$-c=\left(\frac{m+2}{m+1}\right)\frac{c}{2},\ \text{or}\ m=-\frac{4}{3}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(56),$$

and we are substantially conducted to a solution of (38).

With this last value of m, (55) becomes an equation with the same criticoid as (46). Compare Mr. Hargreave's remarks at p. 258 of Vol. VII. of the *Quarterly Journal*; also mine at pp. 243, 244, of Vol. I. of the *Messenger*. Mr. Hargreave treats a form corresponding to (29) by a process of inspection or substitution, and a form corresponding to (46) by obtaining a general value from a particular solution. My discussion connects the latter form with the theory of coresolvents. I may here add that, if we change t in (46) into $-t$, that equation becomes

$$\frac{d^2z}{dt^2} + \left\{ Ae^{2ct} - \frac{c^2}{4} \right\} z = 0 \ldots\ldots\ldots\ldots\ldots\ldots (57),$$

and the origin of the double value of m in (54) and (56) is thus elucidated.

13. By Theorem II., we know that

$$\frac{d^2y}{dx^2} - \frac{n}{2} \cdot \frac{1}{x} \cdot \frac{dy}{dx} + ax^n y = 0 \ldots\ldots\ldots\ldots\ldots\ldots (58)$$

is soluble whatever n may be. If $n = -4$ then (58) becomes

$$\frac{d^2y}{dx^2} + \frac{2}{x} \cdot \frac{dy}{dx} + \frac{a}{x^4} \cdot y = 0 \ldots\ldots\ldots\ldots\ldots\ldots (59),$$

which has the same criticoid as (32), the solubility of which is thus manifested. In general, the criticoid of (58) is

$$ax^n - \frac{n(n+4)}{16} \cdot \frac{1}{x^2}$$

and the solution of (58) may be made to depend upon that of

$$\frac{d^2y}{dx^2} + \left\{ ax^n - \frac{n(n+4)}{16} \cdot \frac{1}{x^2} \right\} y = 0 \ldots\ldots\ldots\ldots\ldots (60),$$

and this again, by exponential substitution, upon that of

$$\frac{dz}{dx} + z^2 + ax_n - \frac{n(n+4)}{16} \cdot \frac{1}{x^2} = 0 \ldots\ldots\ldots\ldots\ldots (61).$$

Assume $$z = \lambda x^{\frac{n}{2}} + \frac{\mu}{x} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (62)$$

and substitute in (61). We have, λ and μ being constant,

$$\frac{\lambda n}{2} \cdot x^{\frac{n}{2}-1} - \frac{\mu}{x^2} + (\lambda^2 + a)\, x^n + 2\lambda\mu x^{\frac{n}{2}-1} + \frac{16\mu^2 - n\,(n+4)}{16x^2} = 0,$$

or $$(\lambda^2 + a)\, x^n + \lambda \left(\frac{n}{2} + 2\mu \right) x^{\frac{n}{2}-1} + \frac{16\mu\,(\mu - 1) - n\,(n+4)}{16x^2} = 0 \ldots. (63),$$

and (63) is satisfied if we make

$$\lambda^2 + a = 0 \text{ and } n + 4\mu = 0 \ldots\ldots\ldots\ldots\ldots\ldots (64, 65).$$

We might of course have solved (58) at once by changing the independent variable.

14. In extension of Ex. 27 of p. 236 of Boole (op. cit.), it may be observed that $x^{n-1} \cdot \dfrac{d^ny}{dx^n}$ is an exact differential coefficient. For

$$\int x^{n-1} \frac{d^ny}{dx^n}\, dx = x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} - (n-1) \int x^{n-2} \frac{d^{n-1}y}{dx^{n-1}}\, dx \ldots\ldots (66).$$

15. In certain earlier steps the theory of coresolvents may be brought into relation with the theory of the comparison of transcendents. Thus, differentiating the equation

$$y^n + x^n = C \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (67),$$

we have, after dividing by n,

$$y^{n-1}dy + x^{n-1}dx = 0 \;\ldots\ldots\ldots\ldots\ldots (68),$$

or, by (67),

$$(C-x^n)^{\frac{n-1}{n}}\,dy + (C-y^n)^{\frac{n-1}{n}}\,dx = 0 \ldots\ldots\ldots\ldots\ldots (69),$$

or, after a division,

$$\frac{dy}{(C-y^n)^{\frac{n-1}{n}}} + \frac{dx}{(C-x^n)^{\frac{n-1}{n}}} = 0 \ldots\ldots\ldots\ldots\ldots (70),$$

of which the primitive is (67). When $n=2$, then (70) becomes

$$\frac{dy}{\sqrt{C-y^2}} + \frac{dx}{\sqrt{C-x^2}} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (71),$$

a familiar form. If $n=\frac{3}{2}$, then (70) becomes

$$\frac{dy}{\sqrt[3]{C-y^{\frac{3}{2}}}} + \frac{dx}{\sqrt[3]{C-x^{\frac{3}{2}}}} = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots (72).$$

16. If we factorially differentiate (58), that is, put it under the form

$$\frac{1}{x^n}\cdot\frac{d^2y}{dx^2} - \frac{n}{2}\cdot\frac{1}{x^{n+1}}\cdot\frac{dy}{dx} + ay = 0 \ldots\ldots\ldots\ldots\ldots\ldots (73),$$

and differentiate, we have, replacing $\frac{dy}{dx}$ by z,

$$\frac{1}{x^n}\cdot\frac{d^2z}{dx^2} - \frac{3n}{2}\cdot\frac{1}{x^{n+1}}\cdot\frac{dz}{dx} + \left\{a + \frac{n(n+1)}{2}\cdot\frac{1}{x^{n+2}}\right\}z = 0 \;\ldots.(73);$$

or, multiplying into x^n,

$$\frac{d^2z}{dx^2} - \frac{3n}{2x}\cdot\frac{dz}{dx} + \left\{ax^n + \frac{n(n+1)}{2}\cdot\frac{1}{x^2}\right\}z = 0 \ldots\ldots\ldots\ldots (74),$$

and the criticoid of this equation is

$$ax^n + \left\{\frac{n(n+1)}{2} - \frac{9n^2}{16} - \frac{3n}{4}\right\}\frac{1}{x^2},$$

which, reduced, becomes

$$ax^n - \frac{n^2+4n}{16},$$

which is equal to the criticoid of (58), and the criticoid has been unchanged by the differentiation.

17. Equation (58) is soluble by change of the independent variable. In ordinary cases we may find equations, the solutions of which depend upon that of a given equation, but which have not the same criticoids. Thus the equations

$$\frac{d^2y}{dx^2} + a^2x^2y = 0,\quad \frac{d^2y}{dx^2} + \frac{2}{x}\cdot\frac{dy}{dx} + a^2x^2y = 0 \ldots\ldots (74, 75)$$

have the same criticoid, viz. a^2x^2. But if we factorially differentiate (74), we have, replacing $\frac{dy}{dx}$ by z, and proceeding in other respects as in Art. 16 for our result

$$\frac{d^2z}{dx^2} - \frac{2}{x}\cdot\frac{dz}{dx} + a^2x^2z = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots (76),$$

whereof the criticoid is $\qquad a^2x^2 - \dfrac{2}{x^2}$;

while if we factorially differentiate (75), and proceed as in the last case, we are conducted to

$$\frac{d^2z}{dx^2} + \left(a^2x^2 - \frac{6}{x^2}\right)z = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots (77),$$

whereof the criticoid is $\qquad a^2x^2 - \dfrac{6}{x^2}$.

Although invariable under factorial substitution, the criticoid is not necessarily so under differentiation or change of the independent variable.

2583. (Proposed by J. J. Walker, M.A.)—Prove that the expression

$$\left\{(a-b)^2(c-d)^2 + (a-c)^2(b-d)^2 + (a-d)^2(b-c)^2\right\}^3$$
$$-8\left\{(a-b)^3(b-c)^3(c-d)^3(d-a)^3 + (a-b)^3(b-d)^3(d-c)^3(c-a)^3 \right.$$
$$\left. + (c-b)^3(b-d)^3(d-a)^3(a-c)^3\right\}$$
$$-24(a-b)^2(a-c)^2(a-d)^2(b-c)^2(b-d)^2(c-d)^2$$

vanishes identically.

Solution by W. H. Laverty; S. Bills; *the* Proposer; *and others.*

Let $\qquad (a-c)^2(b-d)^2 + (a-d)^2(b-c)^2 = \text{P},$

and $\qquad (a-c)(b-d)(a-d)(b-c) = \text{Q},$

then $\qquad \text{P} - 2\text{Q} = (a-b)^2(c-d)^2 = \text{R}$ (suppose),

and $\qquad -(b-c)^3(a-d)^3 + (b-d)^3(a-c)^3 = \text{R}^{\frac{1}{2}}(\text{P}+\text{Q});$

hence the given expression assumes the form

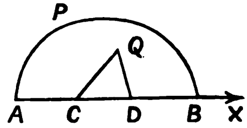
$$\left\{(\text{P}-2\text{Q})+\text{P}\right\}^3 - 8\left\{\text{R}^{\frac{3}{2}}\,\text{R}^{\frac{1}{2}}(\text{P}+\text{Q}) + \text{Q}^3\right\} - 24(\text{P}-2\text{Q})\,\text{Q}^2,$$

which, divided by 8, and simplified, becomes

$$= \text{P}^3 - 3\text{P}^2\text{Q} + 4\text{Q}^3 - \text{R}^2(\text{P}+\text{Q}) = (\text{P}-2\text{Q})^2(\text{P}+\text{Q}) - \text{R}^2(\text{P}+\text{Q}) = 0.$$

2477. (Proposed by T. Savage, M.A.)—If there be described any curve APB upon the chord AB, the average area of all the triangles, formed by joining a point Q taken at random within the area APB and two fixed points C, D in the line AB, is equal to the area of the triangle formed by joining C, D and the centre of gravity of the area APB.

Solution by the Rev. Joseph Wolstenholme, M.A.



Since the curve is all on one side of AB, it is obvious that the average area (Δ say) of the triangle CQD is

$$\Delta = \tfrac{1}{2}\text{CD}\iint y\,dx\,dy \div \iint dx\,dy$$

over the area APB, AB being the axis of x; that is, if (ξ, η) be the coordinates of the centre of gravity G of the area APB, we have $\Delta = \tfrac{1}{2}\text{CD}\,.\,\eta$ = area of triangle CGD.

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